PAULI AND THE PAULI GROUP

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PAULI 1900–1958

1. The Concept of the Pauli Group

$\mathcal{L}(g_i ; \psi_j)$ Lagrangian density

g_i : fundamental parameter

ψ_j : field operator

group G of canonical transformations

$$\psi_j \to \psi_j' \tag{1}$$

$$g_i \to g_i' \tag{2}$$

Assume

$$\mathcal{L}(g_i' ; \psi_j') = \mathcal{L}(g_i ; \psi_j) \tag{3}$$

1) $g_i' = g_i$ Noether's theorem

invariance under G → conservation law

2) $g_i' \neq g_i$

$\mathcal{O}$: observable quantity

$$\mathcal{O}(g_i') = \mathcal{O}(g_i) \tag{4}$$

ex. 1 $\quad \psi$: lepton

$$\mathcal{L}_f(m_i;\psi) = -\bar{\psi}[\gamma_\mu \partial_\mu + (m_1 + i\gamma_5 m_2)]\psi \tag{5}$$

$G = U(1)$

$$\begin{aligned} \psi' &= e^{i\alpha(1-\gamma_5)}\psi, \\ \bar{\psi}' &= \bar{\psi}\, e^{-i\alpha(1+\gamma_5)} \end{aligned} \tag{6}$$

$$\begin{aligned} m_1' &= m_1 \cos 2\alpha - m_2 \sin 2\alpha, \\ m_2' &= m_1 \sin 2\alpha + m_2 \cos 2\alpha. \end{aligned} \tag{7}$$

When $\mathcal{L}_{int}$ is invariant under (6) *)

$$\mathcal{L}(m_1', m_2'; \psi') = \mathcal{L}(m_1, m_2; \psi) \tag{8}$$

Observable quantities depend on m_1 and m_2 through

$$m = (m_1^2 + m_2^2)^{1/2} \tag{9}$$

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*) For instance, (V−A) theory.

ex. 2 Pauli group

$$\mathcal{L}_f = -\bar\psi \gamma_\mu \partial_\mu \psi \qquad \text{massless neutrino} \qquad (10)$$

$G_I = SU(2)$

$$\begin{aligned} \psi \to \psi' &= a\psi + b\gamma_5 C \bar\psi^T, \\ \bar\psi \to \bar\psi' &= a^*\bar\psi - b^* \psi^T C^{-1}\gamma_5 \end{aligned} \qquad (11)$$

with

$$|a|^2 + |b|^2 = 1 \qquad (12)$$

where

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \quad C^T = -C \qquad (13)$$

$G_{II} = U(1)$

$$\begin{aligned} \psi \to \psi' &= e^{i\alpha\gamma_5}\psi, \\ \bar\psi \to \bar\psi' &= \bar\psi\, e^{i\alpha\gamma_5}. \end{aligned} \qquad (14)$$

and

$$G = G_I \times G_{II} = SU(2) \times U(1) \qquad (15)$$

New representation

$$\Psi = \begin{pmatrix} \psi \\ -\gamma_5 C \bar\psi^T \end{pmatrix}, \quad \bar\Psi = (\bar\psi, -\psi^T C^{-1}\gamma_5) \qquad (16)$$

then G_I:

$$\Psi \to \Psi' = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} \Psi, \qquad (11')$$

G_{II}:

$$\Psi \to \Psi' = e^{i\alpha\gamma_5}\Psi \qquad (14')$$

Three generators for SU(2) and one for U(1)

$$M_1 = \frac{1}{2}\int d^3x\,(\psi^\dagger \gamma_5\gamma_4 C \psi^\dagger + \psi C^{-1}\gamma_4\gamma_5\psi),$$

$$M_2 = \frac{1}{2i}\int d^3x\,(\psi^\dagger \gamma_5\gamma_4 C \psi^\dagger - \psi C^{-1}\gamma_4\gamma_5\psi),$$

$$M_3 = \int d^3x\,\psi^\dagger\psi, \tag{17}$$

$$N = \int d^3x\,\psi^\dagger\gamma_5\psi.$$

Commutation Relations

$$\left[\frac{M_i}{2}, \frac{M_j}{2}\right] = i\varepsilon_{ijk}\frac{M_k}{2}, \quad [M_i, N] = 0 \tag{18}$$

In terms of Ψ

$$[\Psi, M_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi, \quad [\Psi, M_2] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\Psi$$

$$[\Psi, M_3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Psi, \quad [\Psi, N] = \gamma_5\Psi \tag{19}$$

and

$$\mathcal{L}_f = -\frac{1}{2}\bar{\Psi}\gamma_\mu\partial_\mu\Psi,$$

$$M_1 = \frac{1}{2}\int d^3x\,\Psi^\dagger\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi, \text{ etc} \tag{20}$$

$$N = \frac{1}{2}\int d^3x\,\Psi^\dagger\gamma_5\Psi,$$

$$\bar{\Psi}\cdot\Psi = 0$$

Beta-decay

$$\mathcal{L}_\beta = \sum_j (\bar{\psi}_p, O_j \psi_n)\left[(\bar{\psi}_e, O_j f_j^\dagger \Psi) + (\bar{\psi}_e, O_j \gamma_5 g_j^\dagger \Psi)\right] + \text{h.c.} \tag{21}$$

where

$$f_j = \begin{pmatrix} f_{j1} \\ f_{j2} \end{pmatrix}, \quad g_j = \begin{pmatrix} g_{j1} \\ g_{j2} \end{pmatrix}, \quad f_j^\dagger = (f_{j1}^*, f_{j2}^*), \quad g_j^\dagger = (g_{j1}^*, g_{j2}^*) \tag{22}$$

Ψ and $f + \gamma_5 g$ transform in the same way under G.

$$\mathcal{L}(f', g'; \Psi') = \mathcal{L}(f, g; \Psi) \tag{23}$$

invariants

$$f_i^\dagger f_j + g_i^\dagger g_j, \quad f_i^\dagger g_j - g_i^\dagger f_j \tag{24}$$

For instance, take the vector coupling

$$G_F^2 = f_V^\dagger f_V + g_V^\dagger g_V. \tag{25}$$

old choice

$$f_V = G_F \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g_V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{26}$$

new choice

$$f_V = \frac{G_F}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g_V = f_V \tag{26'}$$

distinction by measuring the helicity of the electron

Quantum Number $N - M_3$

$$G \supset G_1 = U(1) \times U(1) \tag{27}$$

generated by M_3 and N

$$f_{j1} = g_{j2}, \quad f_{j2} = -g_{j2} \tag{28}$$

G_1

$$\begin{aligned} f'_{j1} &= e^{i(\lambda+\alpha)} f_{j1}, \\ f'_{j2} &= e^{-i(\lambda+\alpha)} f_{j2}. \end{aligned} \tag{29}$$

invariants

$$f^*_{i1} f_{j1}, \quad f^*_{i2} f_{j2}, \quad f_{i1} f_{j2} \tag{30}$$

helicity of electron measured

$$f_{j2} = 0 \tag{31}$$

2. Encounter with Pauli

April 1957 Copenhagen

June 1957 Oberwohlfach

January 1958 New York.

December 1958 deceased