Lecture Note on Intermediate-Energy Neutrino-Nucleus Reactions

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PREFACE

In this note, we wish to clarify nuclear physics aspects in the treatment of neutrino-nucleus reactions in the intermediate-energy region of about 0.5 GeV and up to several GeV, where hadronic treatments are expected to be appropriate. The note is meant to assist neutrino experimentalists for obtaining some relevant information, who have been perhaps trained as particle physicists, yet are facing nuclear physics issues upon analyzing their data. Much of the materials in this note originally appeared in the literature some years ago, but in a rather scattered way. We had to fill in some theoretical steps for a consistent presentation of the treatments and have included them in this note, as they are not readily available. Consequently, some parts of the note may be unnecessarily over-theoretical. At the same time, we have left discussions of the (in particle physics sense) standard neutrino-nucleon reactions brief, except for the parts relevant for nuclear reactions.

To those who are interested in simulation codes of the neutrino-nucleus reactions, perhaps the most pragmatic way to go directly to Subsection IV. F and G of pp 21-24, where we address what we consider to be the most practical computational (Monte Carlo, for example) algorithm; and then go to other parts of the note as needed. Also, the discussions in section V may be useful.

Our original plan was rather ambitious to cover various neutrino reactions. As it stands, it includes mostly the quasi-elastic inclusive (ν, ℓ) and some the semi-inclusive $(\nu, \ell p)$ reactions. The title of the note is thus misnomer. As the reactions expose the major issues in the reactions and we hope to include more materials in new versions, we have decided to leave the title as it is.

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Contents

I. NOTATION

We follow the standard convention:

$$p^2 = g_{\mu\nu} p^\mu p^\nu = g^{\mu\nu} p_\mu p_\nu = p_0^2 - \boldsymbol{p}^2 \tag{1.1}$$

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.2}$$

$$x^\mu = (t, x, y, z) \tag{1.3}$$

$$x_\mu = g_{\mu\nu} x^\nu = (t, -x, -y, -z). \tag{1.4}$$

In this note we assume the neutrinos to be massless. One of the standard conventions for fermions, such as that of Bjorken and Drell, includes the factor of the inverse fermion mass, and thus cannot be used here. We use our own convention for fermions (the neutrinos and nucleons):

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_\lambda \left(a_{\boldsymbol{p}\lambda} u_\lambda(\boldsymbol{p}) e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}\lambda} v_\lambda(\boldsymbol{p}) e^{ip\cdot x} \right) \tag{1.5}$$

$$\{a_{\boldsymbol{p}\lambda}, a^\dagger_{\boldsymbol{q}\lambda'}\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta_{\lambda,\lambda'} \tag{1.6}$$

$$\{b_{\boldsymbol{p}\lambda}, b^\dagger_{\boldsymbol{q}\lambda'}\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta_{\lambda,\lambda'} \tag{1.7}$$

$$|\boldsymbol{p}, \lambda\rangle = \sqrt{2E_p} a^\dagger_{\boldsymbol{p}\lambda} |0\rangle \tag{1.8}$$

$$\langle \boldsymbol{p}', \lambda' | \boldsymbol{p}, \lambda \rangle = 2E_p (2\pi)^3 \delta^3(\boldsymbol{p}' - \boldsymbol{p}) \delta_{\lambda',\lambda} \tag{1.9}$$

$$u_r^\dagger(\boldsymbol{p}) u_s(\boldsymbol{p}) = \delta_{r,s} \tag{1.10}$$

$$\sum_s u_s(\boldsymbol{p}) \overline{u_s}(\boldsymbol{p}) = \frac{\not{p} + m}{2p^0} \tag{1.11}$$

Note $E_p \equiv p^0$ and $\{\psi^\dagger_\alpha(t, \boldsymbol{x}), \psi_\beta(t, \boldsymbol{x}')\} = \delta_{\alpha\beta} \delta^3(\boldsymbol{x} - \boldsymbol{x}')$.

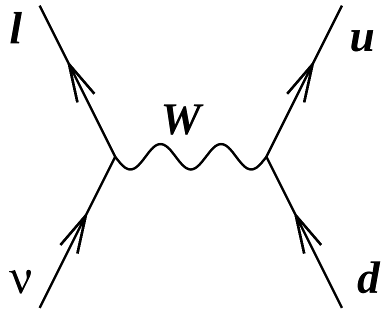

FIG. 1. A charged current reaction of a lepton and quark.

II. INTERACTION HAMILTONIAN

The weak interaction is described by the Glashow-Weinberg-Salam (GWS) theory. Here, we derive an effective hamiltonian in the current-current interaction form from the GWS theory.

In the GWS theory, the interaction hamiltonian density between charged current and weak gauge boson is

$$\mathcal{H}_w = g\left(W_\mu^+ J_\mu^+ + W_\mu^- J_\mu^-\right), \tag{2.1}$$

where $W_\mu^\pm$ is the W gauge boson field and J_μ^+ is the charged current defined by

$$J_\mu^+ = \frac{1}{\sqrt{2}}\left(\overline{\nu}\gamma_\mu \frac{1-\gamma_5}{2}\ell + \overline{u}\gamma_\mu \frac{1-\gamma_5}{2} d\right). \tag{2.2}$$

Here, ν denotes the neutrino field, ℓ the lepton field of either e, μ, or τ, u an up-type quark of either u, c, or t, and d a down-type quark of either d, s, or b. In the followings, we use only the first family for simplicity.

When the momentum transfer is much smaller than weak boson mass, $m_w(= 80GeV)$, the Green function of the weak boson can be approximated as

$$\langle W_\mu(x) W_\nu(y)\rangle = \frac{i g_{\mu\nu}}{m_w^2}\delta^4(x-y). \tag{2.3}$$

The effective hamiltonian for the reaction like Fig. 1 is thus written as

$$\mathcal{H}_w^{eff} = \frac{G}{\sqrt{2}} j_\mu^+ j^{-\mu}, \tag{2.4}$$

where

$$j_\mu^+ = j_{lep_\mu}^+ + j_{had\mu}^+ \tag{2.5}$$

and $G/\sqrt{2} = g^2/(8m_w^2)$. The leptonic current is explicitly given by

$$j_{lep_\mu}^+ = \overline{\nu}\gamma_\mu(1-\gamma_5)e. \tag{2.6}$$

The hadronic current is also described in the same way using quark fields by

$$j_{had\mu}^+ = \overline{u}\gamma_\mu(1-\gamma_5)d. \tag{2.7}$$

III. SINGLE NUCLEON REACTION

We examine the charge-current reaction, $\nu + n \to l^- + p$.

The cross section of the process is

$$d\sigma = \frac{(2\pi)^4}{4|k_1 \cdot p_1|}\frac{d^3\boldsymbol{k}_2}{2k_2^0(2\pi)^3}\frac{d^3\boldsymbol{p}_2}{2p_2^0(2\pi)^3}\overline{\sum_i}\sum_f |\mathcal{M}|^2\delta^4(p_1 + k_1 - p_2 - k_2). \tag{3.1}$$

Here, k_1 denotes the incident neutrino momentum, k_2 the final lepton momentum, p_1 the initial neutron momentum, p_2 the final proton momentum, and $\overline{\sum_i}$ the average over the initial neutron spin-states. We will use m as lepton mass and M as nucleon mass. $\mathcal{M}$ is the invariant amplitude:

$$\mathcal{M} = \frac{G}{\sqrt{2}}\langle l^-, \boldsymbol{k}_2|j^-_{lep_\mu}(0)|\nu, \boldsymbol{k}_1\rangle\langle p, \boldsymbol{p}_2|j^+_{had\mu}(0)|n, \boldsymbol{p}_1\rangle \tag{3.2}$$

The leptonic current becomes

$$\langle l^-, \boldsymbol{k}_2, \lambda|j^-_{lep_\mu}(0)|\nu, \boldsymbol{k}_1\rangle = \sqrt{4k_1k_2^0}\,\overline{u_\lambda}(\boldsymbol{k}_2)\gamma_\mu(1-\gamma_5)u(\boldsymbol{k}_1). \tag{3.3}$$

By summing over the final lepton spin-states, we obtain the leptonic tensor

$$\begin{aligned}\eta_{\mu\nu}(k_1,k_2) &= \sum_\lambda \langle \nu, \boldsymbol{k}_1|j^+_{lep_\mu}(0)|l^-, \boldsymbol{k}_2, \lambda\rangle\langle l^- \boldsymbol{k}_2, \lambda|j^-_{lep_\nu}(0)|\nu, \boldsymbol{k}_1\rangle \\ &= \mathrm{Tr}\left((\not{k}_2 + m_l)\gamma_\mu(1-\gamma_5)\not{k}_1\gamma_\mu(1-\gamma_5)\right) \\ &= 16k_1^\mu k_1^\nu - 8(k_1^\mu q^\nu + q^\mu k_1^\nu) + 4(q^2 - m^2)g^{\mu\nu} + 8i\epsilon^{\mu\nu\alpha\beta}q_\alpha k_{1\beta}\end{aligned} \tag{3.4}$$

where the trace is over spinor indeces, and $q \equiv k_1 - k_2$.

The evaluation of the hadronic currents of the type, $\langle \text{neutron}|j^+_{had\mu}|\text{proton}\rangle$, requires the explicit knowledge of the hadron wave functions in terms of the quarks. The standard method is the phenomenological approach. The hadronic tensor

$$T^{\mu\nu}(p_1,p_2) \equiv \frac{1}{2}\sum_{\lambda_1,\lambda_2}\langle n, \boldsymbol{p}_1, \lambda_1|j^-_{had}{}^\nu(0)|p, \boldsymbol{p}_2, \lambda_2\rangle\langle p, \boldsymbol{p}_2, \lambda_2|j^+_{had}{}^\mu(0)|n, \boldsymbol{p}_1, \lambda_1\rangle \tag{3.5}$$

is expressed in the general form

$$T^{\mu\nu}(p_1,p_2) = -T_1g^{\mu\nu} + T_2p_1^\mu p_1^\nu + T_\alpha q^\mu q^\nu + T_\beta(p_1^\mu q^\nu + q^\mu p_1^\nu) + T_8 i\epsilon^{\mu\nu\alpha\beta}q_\alpha p_{1\beta}, \tag{3.6}$$

where $q \equiv p_2 - p_1$. The coefficients, T's, are functions of q^2 only and will be determined phenomenologically.

Using these tensors, the averaged square of the invariant amplitude is now written as

$$\begin{aligned}\overline{\sum_i}\sum_f|\mathcal{M}|^2 &= \frac{G^2}{2}\eta_{\mu\nu}T^{\mu\nu} \\ &= \frac{G^2}{2}\Big\{-8T_1(q^2-m^2) \\ &\quad +4T_2\left[4p_1\cdot k_1 p_1\cdot k_2 + M^2(q^2-m^2)\right] - 4m^2(q^2-m^2)T_\alpha \\ &\quad -16m^2p_1\cdot k_1(T_\beta + T_8) - 8T_8(k_1+k_2)\cdot p_1(q^2-m^2)\Big\}.\end{aligned} \tag{3.7}$$

A. Single nucleon current

The hadronic currents are not easily expressed in terms of the quark currents because of the composite structure of the hadrons, but the hadronic currents must obey the same fundamental symmetries as the quark currents. The hadronic currents should then have the V-A structure of (vector current) − (axial current),

$$\langle \boldsymbol{p}_2 | j^{+}_{had\,\mu}(0) | \boldsymbol{p}_1 \rangle = j^V_\mu(0) - j^A_\mu(0). \tag{3.8}$$

As the Lorentz vectors, the vector and axial currents are written as

$$j^V_\mu(0) = \sqrt{4E_{p_1}E_{p_2}}\,\overline{u}(\boldsymbol{p}_2)\left[\gamma_\mu F_1(q) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2(q) + q_\mu F_S(q)\right]u(\boldsymbol{p}_1) \tag{3.9}$$

$$j^A_\mu(0) = \sqrt{4E_{p_1}E_{p_2}}\,\overline{u}(\boldsymbol{p}_2)\left[\gamma_\mu\gamma_5 F_A(q) + \gamma_5 q_\mu F_P(q) + i\gamma_5\sigma_{\mu\nu}q^\nu F_T(q)\right]u(\boldsymbol{p}_1). \tag{3.10}$$

Here, u's are nucleon spinors. When the nucleon currents are assumed to be transfered in the same way as the quark current under the charge conjugation and time reverse, the second-class form factors, $F_T(q)$ and $F_S(q)$, vanish and all other form factors become real.

In terms of these form factors, T_i's of the hadronic tensor, $T_{\mu\nu}$, are expressed as

$$T_1 = 2\left[-\frac{q^2}{2}(F_1+F_2)^2 + (2M^2 - \frac{q^2}{2})F_A^2\right] \tag{3.11}$$

$$T_2 = 4\left[F_1^2 - \frac{q^2}{4M^2}F_2^2 + F_A^2\right] \tag{3.12}$$

$$T_\alpha = -2\left[F_1F_2 + \left(2M^2 + \frac{q^2}{2}\right)\frac{F_2^2}{4M^2} + \frac{q^2}{2}F_P^2 - 2MF_PF_A\right] \tag{3.13}$$

$$T_\beta = 2\left[F_1^2 - \frac{q^2}{4M^2}F_2^2 + F_A^2\right] \tag{3.14}$$

$$T_8 = -4F_A(F_1+F_2), \tag{3.15}$$

as functions of q^2 only.

B. Nucleon form factors

The weak (charge-lowering) current, $p \to n$, and its conjugate $n \to p$, together with the electromagnetic current, are believed to form an isospin triplet of conserved vector currents. The form factors in the weak vector current are then directly related to those in the electromagnetic current. The weak-charged current is the incremental or decremental operator of isospin, that is, the current is expressed by the components (τ_1 and τ_2) of the isovector, i.e.,

$$J^\pm_\mu = \overline{U}\Gamma_\mu \frac{\tau_1 \pm i\tau_2}{2}U, \tag{3.16}$$

where $\Gamma_\mu = \gamma_\mu F_1(q) + i\sigma_{\mu\nu}q^\nu F_2(q)/(2M)$ and U is the isodoublet, $(p\ n)^T$. On the other hand, the electromagnetic current is the sum of the isoscalar and the diagonal component (τ_3) of the isovector and then

$$J_\mu^{em} = \frac{1}{2}\overline{U}\Gamma_\mu^S U + \overline{U}\Gamma_\mu \frac{\tau_3}{2} U \tag{3.17}$$

The first term in the right-hand side of the above equation is the isoscalar current and the other is the isovector current. Decomposing the above equation into the proton and neutron currents, we obtain

$$J_\mu^{em} = \overline{p}\frac{1}{2}(\Gamma^S\mu + \Gamma_\mu)p + \overline{n}\frac{1}{2}(\Gamma_\mu^S - \Gamma_\mu)n \tag{3.18}$$

$$= \overline{p}\Gamma_\mu^p p + \overline{n}\Gamma_\mu^n n \tag{3.19}$$

Therefore the form factors of the isovector current becomes

$$F_i(q) = F_i^p(q) - F_i^n(q). \tag{3.20}$$

The form factors in the vector current, $F_1(q)$ and $F_2(q)$, are determined phenomenologically as

$$F_1(q) = \left(1 - \frac{q^2}{4M^2}\right)^{-1}\left[G_E(q) - \frac{q^2}{4M^2}G_M(q)\right] \tag{3.21}$$

$$F_2(q) = \left(1 - \frac{q^2}{4M^2}\right)^{-1}\left[-G_E(q) + G_M(q)\right]. \tag{3.22}$$

Here, $G_E(q)$ and $G_M(q)$ are Sachs form factors,

$$G_E^p(q) = \frac{G_M^p(q)}{\mu_p} = \frac{G_M^n(q)}{\mu_n} = \left(1 - \frac{q^2}{M_V^2}\right)^{-2} \tag{3.23}$$

and $G_E^n(q) = 0$, where the superscripts p and n denote the form factors for the proton and the neutron, respectively. The size parameter is taken to be $M_V^2 = 0.71$ GeV^2. The proton and neutron magnetic moments are $\mu_p \approx 2.793$ and $\mu_n \approx -1.913$ in the unit of the nuclear magneton ($e\hbar/2M_p$ with M_p, the proton mass). We thus obtain the Sachs form factors for the charged current as

$$G_E(q) = \left(1 - \frac{q^2}{M_V^2}\right)^{-2} \tag{3.24}$$

$$G_M(q) = (\mu_p - \mu_n)\left(1 - \frac{q^2}{M_V^2}\right)^{-2}. \tag{3.25}$$

The form facrtors in the axial current, $F_A(q)$ and $F_P(q)$, are related as

$$F_p(q) = \frac{2MF_A(q)}{m_\pi^2 - q^2} \tag{3.26}$$

under PCAC (partially conserved axial current) hypothesis

$$\partial^\mu j_\mu^{Aa} = f_\pi m_\pi^2 \phi_\pi^a \tag{3.27}$$

(with $a = 1$, 2, and 3) together with the assumption that the gauge bosons couple with the nucleon dominantly through the pion exchange in the pseudoscalar channel. $F_A(q)$ is phenomenologically determined as

$$F_A(q) = g_A \left(1 - \frac{q^2}{M_A^2}\right)^{-2} \tag{3.28}$$

with

$$\begin{aligned} g_A &= -1.261 \pm 0.004 \\ M_A &= 1.032 \pm 0.036. \end{aligned} \tag{3.29}$$

This M_A value is the world average quoted in *Physics of Massive Neutrinos* by F. Boehm and P. Vogel (Cambridge University Press, 1992) [1].

$F_P(q)$ contributes the square of the neutrino-nucleon invariant amplitude only through T_α with a coefficient proportional to $(m/M)^2$. $F_P(q)$ contributes to the muon-neutrino cross sections at most about 1 %, and can be neglected. It should not be neglected, however, in the case of the tau-neutrino cross sections.

C. Comments on the currents and form factors

The vector current associated with the nucleon in free space is conserved, as

$$\begin{aligned} q^\mu j_\mu^V(0) &= \sqrt{4E_{p_1}E_{p_2}}\overline{u}(p_2)\left[q^\mu\gamma_\mu F_1(q) + \frac{i\sigma_{\mu\nu}q^\mu q^\nu}{2M}F_2(q)\right]u(p_1) \\ &= \sqrt{4E_{p_1}E_{p_2}}\overline{u}(p_2)\left[(p_2^\mu\gamma_\mu - p_1^\mu\gamma_\mu)F_1(q)\right]u(p_1) \\ &= 0, \end{aligned} \tag{3.30}$$

where we used $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ and the Dirac equation for the nucleon in free-space

$$p_1^\mu\gamma_\mu u(p_1) = Mu(p_1). \tag{3.31}$$

This Dirac equation is not valid for the interacting nucleon, such as the nucleons in nuclei. The vector current is not conserved when the nucleus is described by a system of the interacting nucleons through static potentials. The conservation can be recovered by properly including the meson degrees of freedom, such as the exchange currents.

As well known, the axial current is not conserved in free space:

$$\begin{aligned} q^\mu j_\mu^A(0) &= \sqrt{4E_{p_1}E_{p_2}}\overline{u}(p_2)\left[q^\mu\gamma_\mu\gamma_5 F_A(q) + q^2\gamma_5 F_P(q)\right]u(p_1) \\ &= \overline{u}(p_2)\left[i\sqrt{2}f_\pi(q^2)m_\pi^2\phi_\pi^-\right]u(p_1) \\ &= \frac{2f_\pi(m_\pi^2)m_\pi^2}{q^2 - m_\pi^2}g_{\pi NN}(q^2)\overline{u}(p_2)\gamma_5 u(p_1), \end{aligned} \tag{3.32}$$

where the PCAC hypothesis is applied as an operator relation with $\phi_\pi^- = (\phi^1 - i\phi^2)/\sqrt{2}$.

D. Cross sections

By setting $k_1 = (E_\nu, 0, 0, E_\nu)$ and $p_1 = (M, 0, 0, 0)$ and after carrying out the integration over $\boldsymbol{p}_2$ with the three-dimensional delta function, we obtain

$$d\sigma = \frac{{k_\ell}^2 dk_\ell d\Omega}{64\pi^2 E_\ell E' E_\nu M} \overline{\sum_i} \sum_f |\mathcal{M}|^2 \delta(E_\nu + M - E' - E_\ell), \tag{3.33}$$

where we write $k_2 = (E_\ell, \boldsymbol{k}_\ell)$ with $k_\ell = |\boldsymbol{k}_\ell|$ and $d\Omega = d\phi d\cos\theta$, and also write $p_2 = (E^{'}, \boldsymbol{p}_2)$.

In order to obtain $d\sigma/d\Omega$, we integrate the above expression over k_ℓ by treating (k_ℓ, Ω) as independent variables. That is, we use

$$d\Omega dk_\ell \delta(E_\nu + M - E' - E_\ell) = \frac{d\Omega dk_\ell \delta(k_\ell - ...)}{|\partial(E_\nu + M - E' - E_\ell)/\partial k_\ell|_\Omega}. \tag{3.34}$$

The denominator is evaluated with Ω fixed and is

$$\begin{aligned} \left|\frac{\partial}{\partial k_\ell}(E_\nu + M - E' - E_\ell)\right|_\Omega &= \frac{k_\ell - E_\nu \cos\theta}{E'} + \frac{k_\ell}{E_\ell} \\ &= \frac{k_\ell M}{E_\ell E'}\{1 + \frac{E_\nu}{M}(1 - \frac{E_\ell}{k_\ell}\cos\theta)\} \end{aligned} \tag{3.35}$$

by imposing the δ-function restriction. The differential cross section of $\nu + n \to l^- + p$ in the laboratory coordinate system is then

$$\frac{d\sigma}{d\Omega} = \frac{k_\ell}{64\pi^2 M^2 E_\nu \left[1 + \frac{E_\nu}{M}\left(1 - \frac{E_\ell}{k_\ell}\cos\theta\right)\right]} \overline{\sum_i} \sum_f |\mathcal{M}|^2, \tag{3.36}$$

with

$$\begin{aligned} \overline{\sum_i} \sum_f |\mathcal{M}|^2 = \frac{G^2}{2} \cdot 8 E_\ell E_\nu \Bigg\{ &\left(1 - \frac{k_\ell}{E_\ell}\cos\theta\right)(2T_1 + m^2 T_\alpha) + \left(1 + \frac{k_\ell}{E_\ell}\cos\theta\right) M^2 T_2 \\ &- 2m^2 \frac{M}{E_\ell}(T_\beta + T_8) + 2(E_\nu + E_\ell)\left(1 - \frac{k_\ell}{E_\ell}\cos\theta\right) M T_8 \Bigg\}. \end{aligned} \tag{3.37}$$

The derivation of $d\sigma/dt$ differs, however, from that of $d\sigma/d\Omega$ because (k_ℓ, t), instead of (k_ℓ, Ω), are now treated as the independent variables. Here,

$$\begin{aligned} t &\equiv q^2 \\ &= (k_\ell - k_\nu)^2 \\ &= E_\ell^2 - k_\ell^2 - 2(E_\nu E_\ell - E_\nu k_\ell \cos\theta). \end{aligned} \tag{3.38}$$

In this case, we have with t fixed

$$\left|\frac{\partial}{\partial k_\ell}(E_\nu + M - E' - E_\ell)\right|_t = \frac{k_\ell M}{E_\ell E'}, \tag{3.39}$$

so that

$$
\begin{aligned}
d\Omega dk_\ell \delta(E_\nu + M - E' - E_\ell) &= \left|\frac{\partial\Omega}{\partial t}\right|_{k_\ell} dt dk_\ell \delta(E_\nu + M - E' - E_\ell) \\
&= \frac{\pi}{E_\nu k_\ell} dt dk_\ell \cdot \frac{\delta(k_\ell - ...)}{|\partial(E_\nu + M - E' - E_\ell)/\partial k_\ell|_t} \\
&= \frac{\pi E_\ell E'}{E_\nu k_\ell^2 M} dt dk_\ell \delta(k_\ell - ...). \qquad (3.40)
\end{aligned}
$$

Here, $|\partial\Omega/\partial t|_{k_\ell}$ is the Jacobian arising from the change of the variable

$$
\left|\frac{\partial\Omega}{\partial t}\right|_{k_\ell} = \frac{\pi}{E_\nu k_\ell}, \qquad (3.41)
$$

while the total derivative is

$$
\frac{d\Omega}{dt} = \frac{\pi}{E_\nu k_\ell}\left[1 + \frac{E_\nu}{M}\left(1 - \frac{E_\ell}{k_\ell}\cos\theta\right)\right] \qquad (3.42)
$$

which relates the two forms of the differential cross sections

$$
\frac{d\sigma}{dt} = \frac{d\Omega}{dt} \cdot \frac{d\sigma}{d\Omega}. \qquad (3.43)
$$

We thus have

$$
\frac{d\sigma}{dt} = \frac{1}{64\pi M^2 E_\nu^2} \overline{\sum_i} \sum_f |\mathcal{M}|^2, \qquad (3.44)
$$

where in terms of the Mandelstam variables, t and

$$
\begin{aligned}
s &= (k_\nu + p_1)^2 \qquad (3.45) \\
u &= (k_\nu - p_2)^2, \qquad (3.46)
\end{aligned}
$$

the square of the invariant amplitude is given by

$$
\overline{\sum_i} \sum_f |\mathcal{M}|^2 = \frac{G^2}{2}\Big(A(t) - (s-u)B(t) + (s-u)^2 C(t)\Big) \qquad (3.47)
$$

with $t \equiv q^2$, and

$$
\begin{aligned}
A(t) &= (t - m^2)\left[-8T_1 + (4M^2 - t - m^2)T_2 + 4m^2 T_\beta - 4m^2 T_\alpha\right] \\
&= -4M^2(t - m^2)\left[\left(4 - \frac{t}{M^2}\right)F_A^2 - \left(4 + \frac{t}{M^2}\right)F_1^2 - \frac{t}{M^2}\left(1 + \frac{t}{M^2}\right)F_2^2 \right. \\
&\quad \left. - \frac{4t}{M^2}F_1F_2 - \frac{m^2}{M^2}\left\{(F_1 + F_2)^2 + F_A^2 + tF_p^2 - 4MF_AF_p\right\}\right], \qquad (3.48) \\
B(t) &= -4tT_8 = 16tF_A(F_1 + F_2), \qquad (3.49) \\
C(t) &= T_2 = 4\left(F_1^2 - \frac{t}{4M^2}F_2^2 + F_A^2\right). \qquad (3.50)
\end{aligned}
$$

The factor appearing in $d\sigma/d\Omega$,

$$\frac{1}{1+\frac{E_\nu}{M}(1-\frac{E_\ell}{k_\ell}\cos\theta)}, \tag{3.51}$$

is often called the recoil factor, because it is E_ℓ/E_ν when the lepton mass is neglected and approaches unity as $E_\ell/M \to 0$. We see that the recoil factor has emerged as a consequence of the energy-momentum conservation when the scattering angle (rather than the momentum transfer) is treated as an independent variable.

E. Energy and momentum transfer to the recoiled nucleon

Before we discuss neutrino-nuclear scattering, let us examine the question of what nuclear physics we expect play important roles in the energy of our interest, without going into detailed calculations. As our interest is then of qualitative, or at most semi-quantitative nature, perhaps mere kinematical considerations would be sufficient if appropriately applied. Here, we will make our discussions more definitive by examining the amounts of the energy and momentum transfered to the nucleon by the neutrino-nucleon reaction in *free space*. The discussions are clearly of qualitative nature.

Figures 2 illustrates

$$\langle\omega\rangle = \frac{1}{\sigma(E_\nu)}\int \omega\frac{d\sigma}{d\omega}d\omega \tag{3.52}$$

as a function of E_ν. Similarly, Fig. 3 shows the square-root of

$$\langle\boldsymbol{q}^2\rangle = \frac{1}{\sigma(E_\nu)}\int \boldsymbol{q}^2\frac{d\sigma}{d\boldsymbol{q}^2}d\boldsymbol{q}^2. \tag{3.53}$$

We observe in Fig. 2 that for E_ν greater than a few hundred MeV, the energy transfer becomes larger than 50 MeV, exceeding the deepest (i.e. s-shell) binding energies in (light) nuclei. That is, the recoiled nucleon would be mostly in continuum rather than in low-lying, well-defined nuclear states. In Fig. 3, we see that for the same range of E_ν, the mean momentum transfer becomes several hundred MeV/c, exceeding the (local) Fermi momentum. That is, the physics involved in the process is associated with the inter-nucleon distance or even shorter. Note that the relevant length scale associated with the Fermi momentum, the statistical correlation length, is $\pi/268$ MeV/c $= 2.31$ fm for the nuclear matter.

The observation suggests that upon treating the neutrino reactions at the neutrino energy higher than, say, 0.5 GeV, we should explicitly include the nuclear physics at a short range, such as short-range nuclear correlations at the distance scale of 1 fm or shorter. (In fact, $\pi/1$ fm ≈ 600 MeV/c). We thus expect that the use of a shell model or of a mean-field description alone would be inadequate.

We must be cautious, however. Figures 4 and 5 illustrate the variances of $\langle\omega\rangle$ and $\langle\boldsymbol{q}^2\rangle$, which are shown to be quite large. Therefore, we must properly take account of physics for much wider ranges of the energy and momentum transfer than for what the above discussion suggests. This point becomes significant in the case of the neutrino scattering because

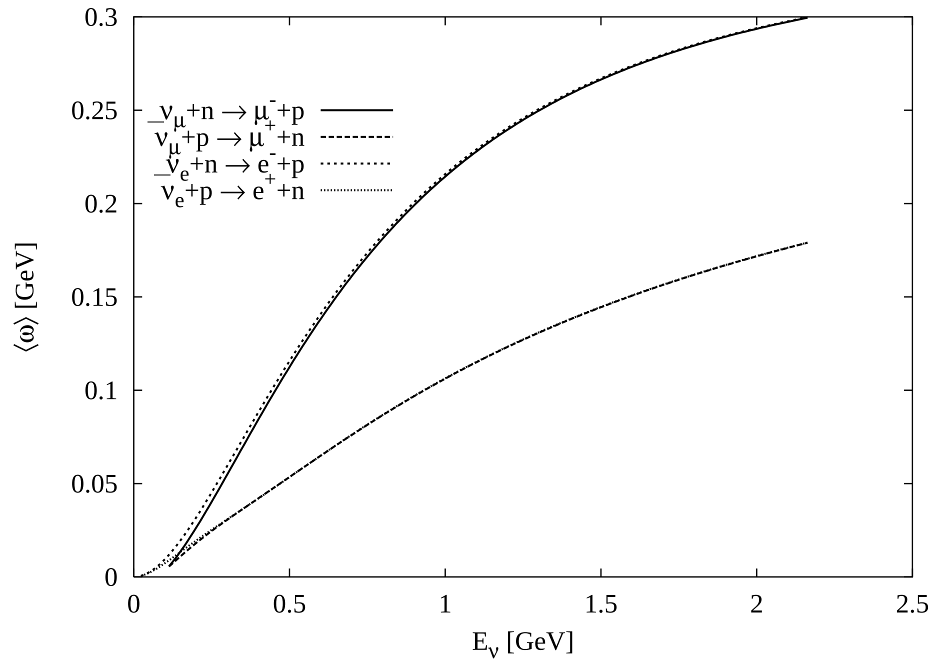

FIG. 2. Average of energy transfer, $\langle\omega\rangle$, weighted by neutrino-nucleon cross section as a function of neutrino energy, E_ν.

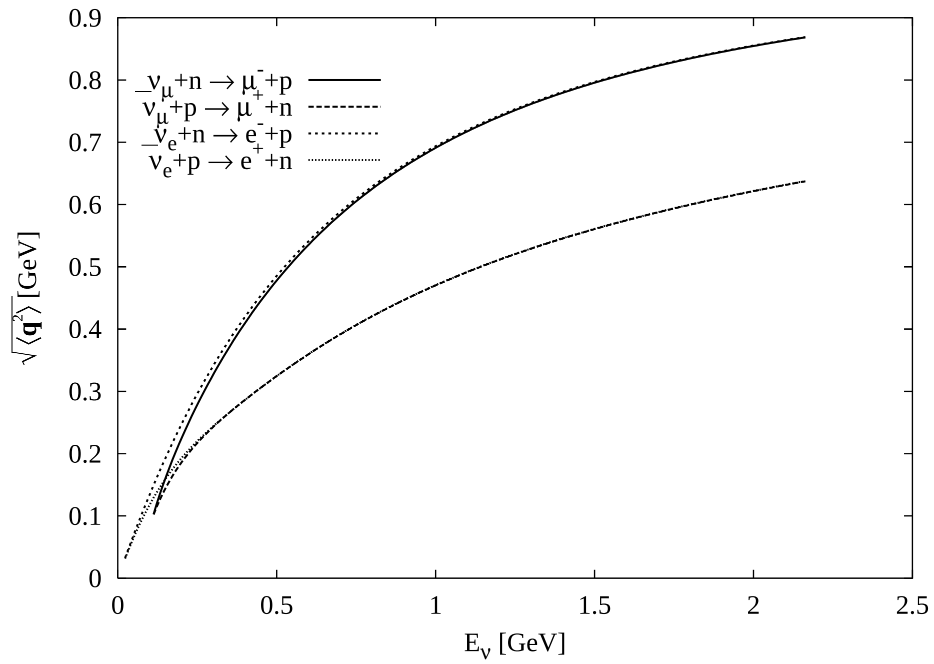

FIG. 3. Average of squared momentum transfer, $\langle \boldsymbol{q}^2 \rangle$, weighted by neutrino-nucleon cross section as a function of neutrino energy, E_ν.

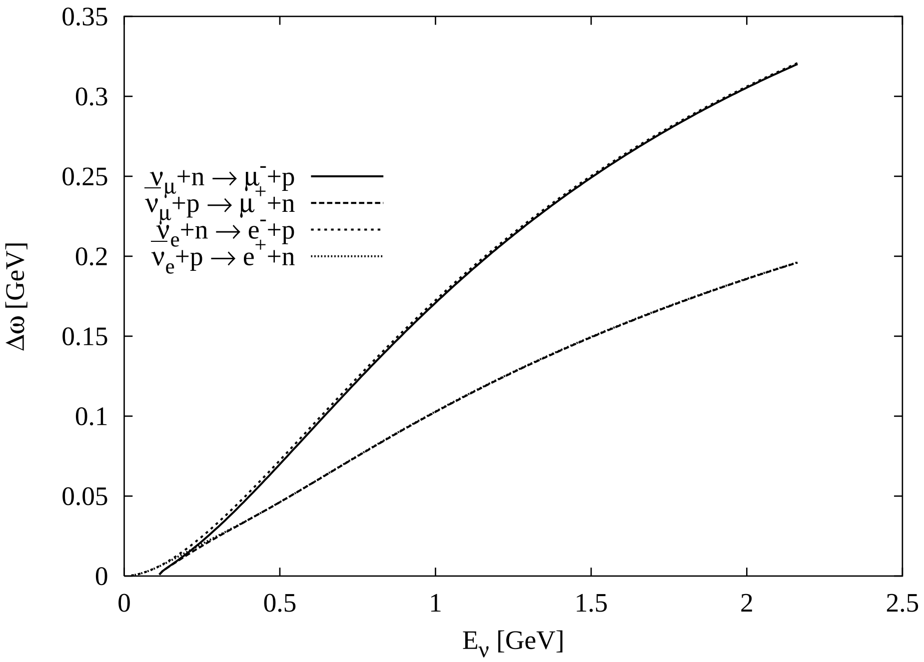

FIG. 4. Variance of energy transfer weighted by neutrino-nucleon cross section as a function of neutrino energy, E_ν.

no monochromatic neutrino beam is available and we often have to examine integrated quantities in comparison to observed data. Note that the forward reaction is observable in the neutrino reactions while it is not the case with electron scattering experiments.

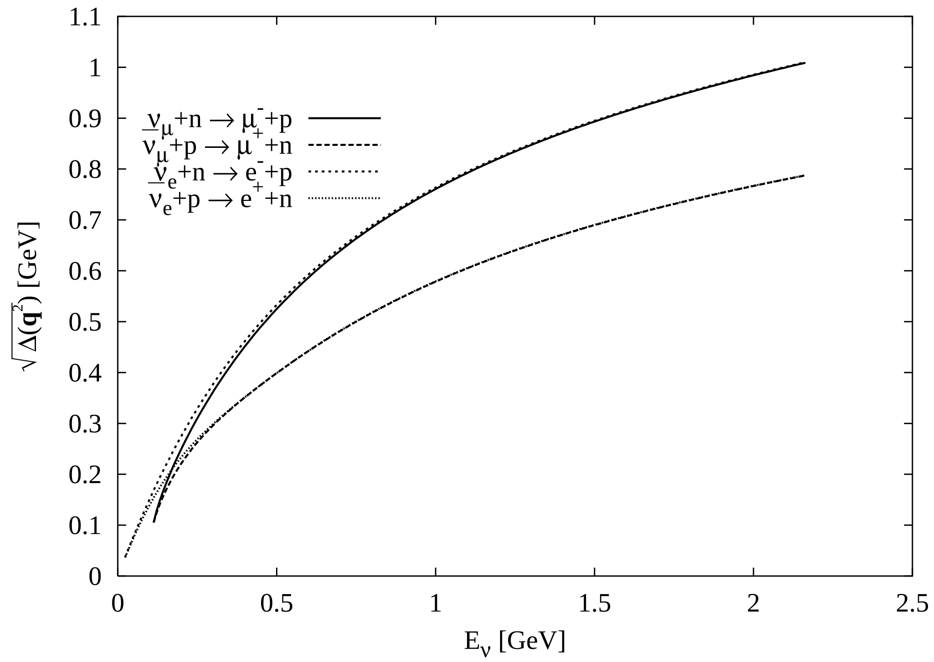

FIG. 5. Variance of squared momentum transfer weighted by neutrino-nucleon cross section as a function of neutrino energy, E_ν.

IV. INCLUSIVE NEUTRINO-NUCLEUS QUASI-ELASTIC SCATTERING; GENERAL FORMULATION

A. Inclusive quasi-elastic cross section

We focus on the inclusive reaction, (ν, ℓ^-), in which neither the ejected nucleon nor the final nuclear state is observed. Figure 6 illustrates the reaction, specifying the notations. We essentially follow R. A. Smith and E. J. Moniz, Nucl. Phys. ***B***43, 605 (1972) [2].

In the previously shown cross section of $k_1 + P_1 \to k_2 + P_2$,

$$d\sigma = \frac{(2\pi)^4}{4|k_1 \cdot P_1|}\frac{d^3\boldsymbol{k}_2}{2k_2^0(2\pi)^3}\frac{d^3\boldsymbol{P}_2}{2P_2^0(2\pi)^3}\overline{\sum_i}\sum_f |\mathcal{M}|^2\delta^4(P_1 + k_1 - P_2 - k_2), \tag{4.1}$$

we set $k_1 = (E_\nu, 0, 0, E_\nu)$, $P_1 \equiv P_A = (M_T, 0, 0, 0)$, $k_2 = (E_\ell, \boldsymbol{k}_\ell)$, and $P_2 \equiv P'_A = (E'_A, \boldsymbol{P}'_A)$.

The inclusive cross section in the laboratory coordinate system is written in terms of the nuclear weak currents $J^\mu(0)$'s as

$$d\sigma = \frac{1}{4E_\nu M_T}\frac{d^3\boldsymbol{k}_\ell}{2E_\ell(2\pi)^3}\frac{G^2}{2}\eta_{\mu\nu}W^{\mu\nu}, \tag{4.2}$$

or

$$\frac{d^2\sigma}{dE_\ell d\Omega} = \frac{E_\ell}{k_\ell}\frac{d^2\sigma}{dk_\ell d\Omega} = \frac{k_\ell}{E_\nu}\frac{G^2}{16M_T(2\pi)^3}\eta_{\mu\nu}W^{\mu\nu}. \tag{4.3}$$

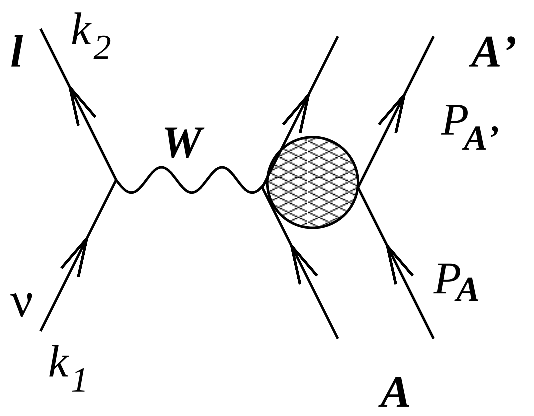

FIG. 6. The Feynman diagram of quasi-elastic neutrino-nucleus reaction.

Here,

$$W^{\mu\nu}(P_A,q)=\overline{\sum_i}\sum_f\int\frac{d^3\boldsymbol{P'}_A}{2P'^0_A(2\pi)^3}\langle P_A,i|J^{(-)\mu}(0)|P'_A,f\rangle\langle P'_A,f|J^{(+)\nu}(0)|P_A,i\rangle$$
$$\times(2\pi)^4\delta^4(P'_A-P_A-q) \tag{4.4}$$

is the hadron (nuclear) tensor, $\eta_{\mu\nu}$ the leptonic tensor as before, and $q=k_2-k_1$ the momentum transfer from the leptons. The hadron (nuclear) tensor is seen to include transitions to all possible, *physical* nuclear states as the intermediate states in the expression, as it should as a description of the inclusive reaction.

$W^{\mu\nu}$ is a second-rank tensor, and can be expressed as a linear combination of the second-rank tensors, $g_{\mu\nu}$ and $\epsilon_{\mu\nu\sigma\tau}$, and of those formed by $P_{A\mu}$ and q_μ:

$$W_{\mu\nu}(P_A,q)/2\pi=-W_1g_{\mu\nu}+\frac{W_2}{M_T^2}P_{A\mu}P_{A\nu}+W_\alpha q_\mu q_\nu+\frac{W_\beta}{M_T}(P_{A\mu}q_\nu+q_\mu P_{A\nu})+\frac{W_8}{M_T}\epsilon_{\mu\nu\sigma\tau}P_A^\sigma q^\tau, \tag{4.5}$$

where the coefficients, W_i's are scalar functions depending on q^2 (as $P_A^2=M_T^2$). In the laboratory system, we have $P_A=(M_T,0,0,0)$ or $P_{A\mu}=M_Tg_{\mu0}$. $W_{\mu\nu}$ is then written as

$$W_{\mu\nu}(q)/2\pi=-W_1g_{\mu\nu}+W_2g_{\mu0}g_{\nu0}+W_\alpha q_\mu q_\nu+W_\beta(g_{\mu0}q_\nu+q_\mu g_{\nu0})+W_8\epsilon_{\mu\nu0\tau}q^\tau. \tag{4.6}$$

By the use of this form of $W_{\mu\nu}$ and the leptonic tensor $\eta_{\mu\nu}$, we can write the neutrino-nucleus reaction cross section as

$$\frac{d^2\sigma}{dE_\ell d\Omega}=\frac{G^2}{16\pi^2}\frac{k_\ell E_\ell}{M_T}\left\{\left(1-\frac{k_\ell}{E_\ell}\cos\theta\right)(2W_1+m^2W_\alpha)+\left(1+\frac{k_\ell}{E_\ell}\cos\theta\right)W_2\right.$$
$$\left.-\frac{2m^2}{E_\ell}(W_\beta+W_8)+2(E_\nu+E_\ell)\left(1-\frac{k_\ell}{E_\ell}\cos\theta\right)W_8\right\}, \tag{4.7}$$

where we denote $\boldsymbol{k}_\ell=(k_\ell,\Omega)$ with $d\Omega=d\phi d\cos\theta$.

When we compare this expression with the previous neutrino-nucleon scattering cross section $d\sigma/d\Omega$, we see the two expressions are similar when we views $W_i \sim T_i$, except for the recoil factor. The two are, however, different in that no k_ℓ integration is carried out in the case of the nuclear cross section, $\frac{d\sigma}{dE_\ell d\Omega}$. As we have seen that the recoil factor emerges upon the k_ℓ integration together with the delta function for the energy conservation. In the nuclear case, the delta function is formulated as a part of $W_{\mu\nu}$, serving to describe propagation of the knocked-out nucleon as we will see below. The recoil factor becomes unity as $E_\nu/M \to 0$, but neglect of E_ν with respect to M_T is not the reason of the missing recoil factor in the nuclear cross section. Note that the final-state nuclear momentum, thus the energy, is integrated out in the nuclear case.

B. Nuclear weak current

The incident neutrino interacts with the nucleons in the nucleus, and thus we write the nuclear weak current, to which the leptonic tensor couples, as the sum of the nucleon weak currents,

$$J^{(+)\mu}(0) = \sum_{\lambda_1,\lambda_2} \int \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2}{(2\pi)^6 2\sqrt{p_1^0 p_2^0}} a^\dagger_{\boldsymbol{p}_2\lambda_2} \langle \boldsymbol{p}_2, \lambda_2 | j^{(+)\mu}(0) | \boldsymbol{p}_1, \lambda_1 \rangle a_{\boldsymbol{p}_1\lambda_1} \tag{4.8}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the nucleon momenta, $a^\dagger_{\boldsymbol{p}_1\lambda_1}$ is the creation operator of the nucleon (state) with the momentum $\boldsymbol{p}_1$ and the spin-state λ_1, and $a_{\boldsymbol{p}_2\lambda_2}$ is the annihilation operator similarly defined. Though these expressions appear to be quite reasonable, we are taking only the nucleon degree of freedom in the nucleus, to which the leptonic current couples. That is,

- the coupling of the leptonic current to nuclear exchange currents is neglected.

Perhaps this coupling could be effectively included up to some degree, by suitable modification to the nucleon coupling. But this is not the standard practice.

We first write $W_{\mu\nu}$ as

$$W^{\mu\nu} = \overline{\sum_i} (2\pi)^4 \langle P_A, i | J^{(-)\mu}(0) \delta^4(\hat{P} - P_A - q) J^{(+)\nu}(0) | P_A, i \rangle, \tag{4.9}$$

where $\hat{P} = (\hat{H}, \hat{\boldsymbol{P}})$ is the operator for the (nuclear) four momentum. Note that

$$\sum_f \int \frac{d^3\boldsymbol{P'}_A}{2P'^0_A (2\pi)^3} |P'_A, f\rangle\langle P'_A, f| = 1 \tag{4.10}$$

in our convention. We now use the above expression of the nuclear current and obtain

$$\begin{aligned} W^{\mu\nu} = \overline{\sum_i} \sum_{\lambda_1\lambda_2\lambda'_1\lambda'_2} (2\pi)^4 \int & \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^{12} 4\sqrt{p_1^0 p_2^0 p'^0_1 p'^0_2}} \\ & \times \langle P_A, i | a^\dagger_{\boldsymbol{p}'_1\lambda'_1} a_{\boldsymbol{p}'_2\lambda'_2} \delta^4(\hat{P} - P_A - q) | a^\dagger_{\boldsymbol{p}_2\lambda_2} a_{\boldsymbol{p}_1\lambda_1} | P_A, i \rangle \\ & \times \langle \boldsymbol{p}'_1, \lambda'_1 | j^{(-)\mu}(0) | \boldsymbol{p}'_2, \lambda'_2 \rangle \langle \boldsymbol{p}_2, \lambda_2 | j^{(+)\nu}(0) | \boldsymbol{p}_1, \lambda_1 \rangle \end{aligned} \tag{4.11}$$

$$\equiv \int \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^8 4\sqrt{p_1^0 p_2^0 {p'}_1^0 {p'}_2^0}}$$
$$\times \overline{\sum_i} F({p'}_1\lambda'_1, {p'}_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) \sum_{\lambda_1\lambda_2\lambda'_1\lambda'_2} T^{\mu\nu}({p'}_1\lambda'_1, {p'}_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) \tag{4.12}$$

Here, $W_{\mu\nu}$ consists of the two factors, F and T. F describes the nuclear response to the neutrino interaction,

$$\begin{aligned}
&F({p'}_1\lambda'_1, {p'}_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) \\
&= \langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1} a_{\boldsymbol{p'}_2\lambda'_2} \delta^3(\hat{\boldsymbol{P}} - \boldsymbol{P}_A - \boldsymbol{q}) \delta(\hat{H} - P_A^0 - \omega) a^\dagger_{\boldsymbol{p}_2\lambda_2} a_{\boldsymbol{p}_1\lambda_1} |P_A, i\rangle \\
&= \delta^3(\boldsymbol{p}'_2 - \boldsymbol{p}'_1 - \boldsymbol{q}) \langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1} a_{\boldsymbol{p'}_2\lambda'_2} \delta(\hat{H} - P_A^0 - \omega) a^\dagger_{\boldsymbol{p}_2\lambda_2} a_{\boldsymbol{p}_1\lambda_1} |P_A, i\rangle \\
&= \delta^3(\boldsymbol{p}'_2 - \boldsymbol{p}'_1 - \boldsymbol{q}) \mathrm{Im} \frac{1}{\pi i} \int_0^\infty dt e^{i\omega t} \langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1}(t) a_{\boldsymbol{p'}_2\lambda'_2}(t) a^\dagger_{\boldsymbol{p}_2\lambda_2}(0) a_{\boldsymbol{p}_1\lambda_1}(0) |P_A, i\rangle,
\end{aligned} \tag{4.13}$$

where the t-dependence of the creation and annihilation operators is introduced through

$$a_{\boldsymbol{p}\lambda}(t) = e^{i\hat{H}t} a_{\boldsymbol{p}\lambda} e^{-i\hat{H}t} = e^{i(\hat{H} - P_A^0)t} a_{\boldsymbol{p}\lambda} e^{-i(\hat{H} - P_A^0)t} \tag{4.14}$$

with the nuclear Hamiltonian operator, $\hat{H}$, including the nuclear rest mass. In the above equations, we used

$$\langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1} a_{\boldsymbol{p'}_2\lambda'_2} \hat{\boldsymbol{P}} = \langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1} a_{\boldsymbol{p'}_2\lambda'_2} (\boldsymbol{p}'_2 - \boldsymbol{p}'_1 + \boldsymbol{P}_A), \tag{4.15}$$

or alternatively we could use

$$\hat{\boldsymbol{P}} a^\dagger_{\boldsymbol{p}_2\lambda_2} a_{\boldsymbol{p}_1\lambda_1} |P_A, i\rangle = (\boldsymbol{p}_2 - \boldsymbol{p}_1 + \boldsymbol{P}_A) a^\dagger_{\boldsymbol{p}_2\lambda_2} a_{\boldsymbol{p}_1\lambda_1} |P_A, i\rangle. \tag{4.16}$$

Note that $\boldsymbol{P}_A = 0$ in our case.

We thus have

$$\begin{aligned}
W^{\mu\nu} = &\sum_{\lambda_1\lambda_2\lambda'_1\lambda'_2} \int \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 d^3\boldsymbol{p}'_1}{(2\pi)^8 4\sqrt{p_1^0 p_2^0 {p'}_1^0 {p'}_2^0}} \\
&\times \overline{\sum_i} \mathrm{Im} \frac{1}{\pi i} \int_0^\infty dt e^{i\omega t} \langle P_A, i| a^\dagger_{\boldsymbol{p'}_1\lambda'_1}(t) a_{\boldsymbol{p'}_1 + \boldsymbol{q}\lambda'_2}(t) a^\dagger_{\boldsymbol{p}_2\lambda_2}(0) a_{\boldsymbol{p}_1\lambda_1}(0) |P_A, i\rangle \\
&\times T^{\mu\nu}({p'}_1\lambda'_1, ({p'}_1 + q)\lambda'_2, p_1\lambda_1, p_2\lambda_2).
\end{aligned} \tag{4.17}$$

T is the nucleon tensor,

$$T^{\mu\nu}({p'}_1\lambda'_1, ({p'}_1 + q)\lambda'_2, p_1\lambda_1, p_2\lambda_2) = \langle \boldsymbol{p'}_1, {\lambda'}_1 | j^{(-)\mu} | \boldsymbol{p'}_1 + \boldsymbol{q}, {\lambda'}_2 \rangle \langle \boldsymbol{p}_2, \lambda_2 | j^{(+)\nu} | \boldsymbol{p}_1, \lambda_1 \rangle. \tag{4.18}$$

Combined with $\eta_{\mu\nu}$, it forms $T^{\mu\nu}\eta_{\mu\nu}$, a quantity similar to the neutrino-nucleon cross section, but depending on nucleon momenta, p_1, p_2, and p'_1, and also on the momentum transfer, $q = k_2 - k_1$.

C. The use of the single-particle nuclear wave function

The approximation usually made at this point is:

- The nuclear wave functions in the initial and final states are completely specified in terms of the (spatial or momentum) coordinates of each nucleon, that is, in terms of single-particle

  wave functions (including the nucleons in continuum when necessary.)

The ground-state nucleus is described in the single-particle states of the nucleons, $(j_1, j_2, \cdots, j_A)$, which also specify the spin state of each nucleon:

$$\begin{aligned} |P_A, i\rangle_S &= \int \frac{\Pi_{m=1}^{A} d^3\bar{\boldsymbol{p}}_m}{\Pi_{m=1}^{A}\{(2\pi)^3 2\bar{p}_m^0\}} |\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \bar{\boldsymbol{p}}_A j_A\rangle\langle\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \bar{\boldsymbol{p}}_A j_A|P_A, i\rangle \\ &= \int \frac{\Pi_{m=1}^{A} d^3\bar{\boldsymbol{p}}_m}{\Pi_{m=1}^{A}\{(2\pi)^3 2\bar{p}_m^0 n_{j_m}!\}} a^\dagger_{\bar{\boldsymbol{p}}_1 j_1} a^\dagger_{\bar{\boldsymbol{p}}_2 j_2} \cdots a^\dagger_{\bar{\boldsymbol{p}}_A j_A}|0\rangle \Psi_i(\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \bar{\boldsymbol{p}}_A j_A; P_A), \end{aligned} \tag{4.19}$$

where n_{j_m} is the occupation number of the state $|j_m\rangle$ in $|j_1, j_2, \cdots, j_A\rangle$, and

$$\Psi_i(\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \bar{\boldsymbol{p}}_A j_A; P_A) = \langle\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \bar{\boldsymbol{p}}_A j_A|P_A, i\rangle \tag{4.20}$$

is the single-particle ground-state nuclear wave function. Using this $|P_A, i\rangle$, we carry out straightforward but somewhat lengthy algebra with the creation and annihilation operators, and obtain

$$\begin{aligned} &F(p'_1\lambda'_1, p'_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) = \delta^3(\boldsymbol{p}'_2 - \boldsymbol{p}'_1 - \boldsymbol{q}) \\ &\times \mathrm{Im}\frac{1}{\pi i}\int_0^\infty dt e^{i\omega t}{}_S\langle(\boldsymbol{p}'_1\lambda'_1)^{-1}P_A, i|e^{-i(\hat{H}-P_A^0)} a_{\boldsymbol{p}'_2\lambda'_2}(t) a^\dagger_{\boldsymbol{p}_2\lambda_2}(0)|(\boldsymbol{p}_1\lambda_1)^{-1}P_A, i\rangle_S, \end{aligned} \tag{4.21}$$

where $|(\boldsymbol{p}_1\lambda_1)^{-1}P_A, i\rangle_S$ is the single-particle nucleus in which one of the nucleons is removed with the momentum p_1 and the spin λ_1, that is, a hole is created in the single-particle ground-state nucleus with $\boldsymbol{p}_1$ and λ_1:

$$\begin{aligned} |(\boldsymbol{p}_1\lambda_1)^{-1}P_A, i\rangle_S = \int &\frac{\Pi'^{A}_{m=1} d^3\bar{\boldsymbol{p}}_m}{\Pi'^{A}_{m=1}\{(2\pi)^3 2p_m^0 n_{j_m}\}} a^\dagger_{\bar{\boldsymbol{p}}_1 j_1} a^\dagger_{\bar{\boldsymbol{p}}_2 j_2} \cdots a^\dagger_{\bar{\boldsymbol{p}}_A j_A}|0\rangle \\ &\times \Psi_i(\bar{\boldsymbol{p}}_1 j_1, \bar{\boldsymbol{p}}_2 j_2, \cdots, \boldsymbol{p}_1 j(\lambda_1), \cdots, \bar{\boldsymbol{p}}_A j_A; P_A), \end{aligned} \tag{4.22}$$

where Π' excludes the factor associated with the single-particle hole state, and $j(\lambda_1)$ implies the appropriate matching of the hole angular momentum j and the spin quantum number λ_1.

The hadron (nuclear) tensor is

$$\begin{aligned} W^{\mu\nu} = &\sum_{\lambda_1\lambda_2\lambda'_1\lambda'_2} \int \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 d^3\boldsymbol{p}'_1}{(2\pi)^8 4\sqrt{p_1^0 p_2^0 p'^0_1 p'^0_2}} \\ &\times \overline{\sum_i} \mathrm{Im}\frac{1}{\pi i}\int_0^\infty dt e^{i\omega t}{}_S\langle(\boldsymbol{p}'_1\lambda'_1)^{-1}P_A, i|e^{-i(\hat{H}-P_A^0)t} a_{\boldsymbol{p}'_1+\boldsymbol{q}\lambda'_2}(t) a^\dagger_{\boldsymbol{p}_2\lambda_2}(0)|P_A, i(\boldsymbol{p}_1\lambda_1)^{-1}\rangle_S \\ &\times T^{\mu\nu}(p'_1\lambda'_1, p'_1 + q\lambda'_2, p_1\lambda_1, p_2\lambda_2). \end{aligned} \tag{4.23}$$

The neutrino-nucleus reaction cross section is then proportional to $\eta_{\mu\nu}T^{\mu\nu}$, appearing in the integrand. $\eta_{\mu\nu}T^{\mu\nu}$ forms the neutrino-nucleon invariant amplitude, but it depends on three off-the-mass-shell nucleon momenta because the nucleons are bound in the single-particle states. The common practice (also in electron and muon scattering) at this point is

- the use of the nucleon currents in the nucleus with the on-(the-mass-)shell kinematics.

This practice artificially reserves the apparent vector current conservation in nuclei, at least for now.

$a^\dagger_{\boldsymbol{p}_2\lambda_2}(0)$ creates a nucleon with the momentum $\boldsymbol{p}_2$ above the occupied single-particle nuclear states at $t=0$, and $a_{\boldsymbol{p'}_1+\boldsymbol{q}\lambda'_2}(t)$ annihilates it with the momentum $\boldsymbol{p}'_1+\boldsymbol{q}$ at time t. The propagation of the nucleon takes place through single-particle states in continuum. For a large $\boldsymbol{q}$, an approximation is sometimes made, in which the continuum states are replaced by free particles (plane waves). This yields

$$
\begin{aligned}
&\overline{\sum_i} F(p'_1\lambda'_1, p'_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) \\
&= \overline{\sum_i}\mathrm{Im}\frac{1}{\pi i}\int_0^\infty dte^{i\omega t}{}_S\langle(\boldsymbol{p}'_1\lambda'_1)^{-1}P_A, i|e^{-i(\hat{H}-P_A^0)t}a_{\boldsymbol{p'}_1+\boldsymbol{q}\lambda'_2}(t)a^\dagger_{\boldsymbol{p}_2\lambda_2}(0)|P_A, i(\boldsymbol{p}_1\lambda_1)^{-1}\rangle_S \\
&\approx \delta^3(\boldsymbol{p}'_1+\boldsymbol{q}-\boldsymbol{p}_2)\delta_{\lambda'_2\lambda_2}\overline{\sum_i}\mathrm{Im}\frac{1}{\pi i}\int_0^\infty dte^{i\omega t}{}_S\langle(\boldsymbol{p}'_1\lambda'_1)^{-1}P_A, i|e^{-i(\hat{H}-P_A^0)t}|P_A, i(\boldsymbol{p}_1\lambda_1)^{-1}\rangle_S \\
&= \delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)\delta_{\lambda'_1\lambda_1}\delta^3(\boldsymbol{p}'_1+\boldsymbol{q}-\boldsymbol{p}_2)\delta_{\lambda'_2\lambda_2}\overline{\sum_i}{}_S\langle P_A, i|a^\dagger_{\boldsymbol{p}_1\lambda_1}\delta(\hat{H}-P_A^0-\omega)a_{\boldsymbol{p}_1\lambda_1}|P_A, i\rangle_S. \qquad (4.24)
\end{aligned}
$$

Together with the on-the-mass-shell kinematics, this procedure then enables us to use the on-mass-shell neutrino-nucleon cross section. The procedure neglects final-state interactions of the knocked-out nucleon, and is called plane-wave impulse approximation (PWIA). The final-state interactions (strictly speaking, the distortion due to being in the single-particle continuum in this frame work) is sometimes included approximately, together with the on-the-mass-shell kinematics. The procedure is called distorted-wave impulse approximation (DWIA). We will not discuss these procedures further for now.

D. Factorization of two-particle Green's function

For high-energy electroweak reactions, the high-momentum nucleons in the nucleus should be also accounted for, as they are expected to play an important role. The high-momentum nucleons emerge, for example, because of nuclear correlations, and thus the above single-particle description would not be adequate for the high-energy reactions. At the same time, we expect that contributions at a large-momentum transfer become more appreciable at high energies, and semi-classical approaches generally become more valid. Accordingly, we make the following approximation with the nuclear response,

$$
\begin{aligned}
&\overline{\sum_i} F(p'_1\lambda'_1, p'_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) \\
&= \delta^3(\boldsymbol{p}'_2-\boldsymbol{p}'_1-\boldsymbol{q})\overline{\sum_i}\mathrm{Im}\frac{1}{\pi i}\int_0^\infty dte^{i\omega t}\langle P_A, i|a^\dagger_{\boldsymbol{p'}_1\lambda'_1}(t)a_{\boldsymbol{p'}_2\lambda'_2}(t)a^\dagger_{\boldsymbol{p}_2\lambda_2}(0)a_{\boldsymbol{p}_1\lambda_1}(0)|P_A, i\rangle \\
&\approx \delta^3(\boldsymbol{p}'_2-\boldsymbol{p}'_1-\boldsymbol{q})\mathrm{Im}\frac{1}{\pi i}\int_0^\infty \frac{dt}{2P_A^0V}e^{i\omega t}\langle P_A|a^\dagger_{\boldsymbol{p'}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle\langle P_A|a_{\boldsymbol{p'}_2}(t)a^\dagger_{\boldsymbol{p}_2}(0)|P_A\rangle, \qquad (4.25)
\end{aligned}
$$

by expressing the two-particle Green's function as a product of two single-particle Green's functions. The single-particle Green's functions are assumed to have no dependence on the nucleon spins and on the initial nuclear state. The spatial normalization, V, appears here as seen in the nuclear state-vector normalization,

$$\langle P_A|P_A\rangle = 2P_A^0(2\pi)^3\delta^3(\boldsymbol{P}=0) \equiv 2P_A^0 V. \tag{4.26}$$

Now, $a_{\boldsymbol{p}_1}|P_A\rangle$ is a momentum eigenstate, and the momentum operator commutes with the total Hamiltonian operator. $\langle P_A|a^\dagger_{\boldsymbol{p'}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle$ is thus proportional to $\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)$, and we write

$$\langle P_A|a^\dagger_{\boldsymbol{p'}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle = \langle P_A|a^\dagger_{\boldsymbol{p'}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle(2\pi)^3\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)/V. \tag{4.27}$$

Therefore,

$$\begin{aligned}
&\overline{\sum_i} F(p'_1\lambda'_1, p'_2\lambda'_2, p_1\lambda_1, p_2\lambda_2) = \frac{(2\pi)^6}{2P_A^0V^3}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_2)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}'_1-\boldsymbol{q})\\
&\times \mathrm{Im}\frac{1}{\pi i}\int_0^\infty dt e^{i\omega t}\langle P_A|a^\dagger_{\boldsymbol{p}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle\cdot\langle P_A|a_{\boldsymbol{p}_2}(t)a^\dagger_{\boldsymbol{p}_2}(0)|P_A\rangle\\
&= \delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_2)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}'_1-\boldsymbol{q})\\
&\times\frac{2(2\pi)^6P_A^0}{V}\int d\omega' P_h(\boldsymbol{p}_1,\omega')P_p(\boldsymbol{p}_1+\boldsymbol{q},\omega-\omega').
\end{aligned} \tag{4.28}$$

Here, we have introduced the hole and particle spectral functions,

$$\begin{aligned}
P_h(\boldsymbol{p},\omega) &= \mathrm{Im}\frac{1}{\pi i}\int_0^\infty dt e^{i\omega t}\langle P_A|a^\dagger_{\boldsymbol{p}_1}(t)a_{\boldsymbol{p}_1}(0)|P_A\rangle\\
&= \frac{1}{2P_A^0V}\langle P_A|a^\dagger_{\boldsymbol{p}}\delta(\hat{H}-P_A^0-\omega)a_{\boldsymbol{p}}|P_A\rangle\\
P_p(\boldsymbol{p},\omega) &= \mathrm{Im}\frac{1}{\pi i}\int_0^\infty dt e^{i\omega t}\langle P_A|a_{\boldsymbol{p}_1}(t)a^\dagger_{\boldsymbol{p}_1}(0)|P_A\rangle\\
&= \frac{1}{2P_A^0V}\langle P_A|a_{\boldsymbol{p}}\delta(\hat{H}-P_A^0-\omega)a^\dagger_{\boldsymbol{p}}|P_A\rangle,
\end{aligned} \tag{4.29}$$

respectively. Note that the product of the real parts of the single-particle Green's functions do not contributes to the integral over t.

We thus have

$$W^{\mu\nu} = \int\frac{d^3\boldsymbol{p}_1 d\omega' P_A^0}{(2\pi)^2Vp_1^0(p_1^0+\omega)}P_h(\boldsymbol{p}_1,\omega')P_p(\boldsymbol{p}_1+\boldsymbol{q},\omega-\omega')T^{\mu\nu}(p_1,p_1+q), \tag{4.30}$$

where, as p_1 is now taken to be on-the-mass-shell,

$$T^{\mu\nu}(\boldsymbol{p}_1,q) \equiv \frac{1}{2}\sum_{\lambda_1\lambda_2}T^{\mu\nu}(p_1\lambda_1, p_1+q\lambda_2, p_1\lambda_1, p_1+q\lambda_2). \tag{4.31}$$

E. Integrated factorization and Linear response

We then have the structure

$$W^{\mu\nu}(q) = \int \frac{d^3\boldsymbol{p}}{2\pi P_A^0} F(\boldsymbol{p}, q) T^{\mu\nu}(\boldsymbol{p}, q), \tag{4.32}$$

where $F(\boldsymbol{p}, q)$ describes the nuclear response, independent of the neutrino-nucleon reaction that is represented the coupling of the leptonic tensors. It has the form either introduced in the factorization approximation of the two-particle Green's function,

$$F(\boldsymbol{p}, q) = \frac{M_T}{(2\pi)^3 V} \int \frac{d\omega'}{p^0(p^0+\omega)} P_h(\boldsymbol{p}, \omega') P_p(\boldsymbol{p}+\boldsymbol{q}, \omega-\omega'), \tag{4.33}$$

or the corresponding one appearing in PWIA (or DWIA) by the use of the single-particle nuclear wave function. We thus have

$$\frac{d^2\sigma}{dE_\ell d\Omega} = \frac{k_\ell}{8(2\pi)^4 M_T^2 E_\nu} \int d^3\boldsymbol{p} F(\boldsymbol{p}, q) \frac{G^2}{2} \eta_{\mu\nu} T^{\mu\nu}(\boldsymbol{p}, q; E_\nu). \tag{4.34}$$

Here, $G^2 \eta_{\mu\nu} T^{\mu\nu}/2$ is a Lorentz-invariant quantity,

$$\frac{G^2}{2} \eta_{\mu\nu} T^{\mu\nu}(\boldsymbol{p}, q; E_\nu) = \overline{\sum_i} \sum_f |\mathcal{M}|^2(s, t, u), \tag{4.35}$$

depending only on the Lorentz-invariant (Mandelstam) variables, (s, t, u). and is related to $d\sigma/dt$,

$$\frac{G^2}{2} \eta_{\mu\nu} T^{\mu\nu}(\boldsymbol{p}, q; E_\nu) \propto \frac{d\sigma}{dt}(\boldsymbol{p}, q; E_\nu), \tag{4.36}$$

as in the case of the target nucleon at rest,

$$\frac{G^2}{2} \eta_{\mu\nu} T^{\mu\nu}(\boldsymbol{p}=0, q; E_\nu) = 32\pi M^2 E_\nu^2 \left(\frac{d\sigma}{dt}\right), \tag{4.37}$$

shown in Section III.

We thus have

$$\frac{\delta(d\sigma/dE_\ell d\Omega)}{\delta(d\sigma/dt)(\boldsymbol{p}, q)} \propto F(\boldsymbol{p}, q). \tag{4.38}$$

The inclusive neutrino-nucleus cross section, $d\sigma/dk_\ell d\Omega$, is thus expressed as a product of the neutrino-nucleon cross section and the nuclear response to it. This is the general expression of the linear response theory, and F is an example of the response functions appearing in condensed matter and nuclear physics. Note, however, that the above expression is somewhat schematic, as we have been cavalier with the treatment of the delta function representing the energy conservation. We will treat it more carefully in the following subsection.

F. Computational algorithm and off-shell kinematics

The above equation

$$\frac{d^2\sigma}{dE_\ell d\Omega} = \frac{k_\ell}{8(2\pi)^4 M_T^2 E_\nu} \int d^3\boldsymbol{p} F(\boldsymbol{p},q) \frac{G^2}{2} \eta_{\mu\nu} T^{\mu\nu}(\boldsymbol{p},q;E_\nu), \tag{4.39}$$

where

$$\begin{aligned} \frac{G^2}{2}\eta_{\mu\nu}T^{\mu\nu}(\boldsymbol{p},q;E_\nu) &= \overline{\sum_i}\sum_f |\mathcal{M}|^2(s,t,u) \\ &= \frac{G^2}{2}\Big(A(t) - (s-u)B(t) + (s-u)^2 C(t)\Big), \end{aligned} \tag{4.40}$$

expresses $d^2\sigma/(dE_\ell d\Omega)$ as a nucleon-momentum integral of the product of the nuclear response and of the leptonic and nucleonic current coupling. This equation provides a convenient means to compute $d^2\sigma/(dE_\ell d\Omega)$ in various ways including Monte Carlo methods. $G^2\eta_{\mu\nu}T^{\mu\nu}/2$ is a Lorentz-invariant quantity, and is evaluated simply using s, t, and u determined from $E_\nu, \mathbf{p}$, and q.

$F(\boldsymbol{p},q)$ can be of Fermi gas model (see Subsection V. A) of a more elaborate description of the nucleus, or of the one including the final-state interactions after the initial neutrino-nucleon reaction. Eitherway, once $F(\boldsymbol{p},q)$ is chosen, $d^2\sigma/(dE_\ell d\Omega)$ is calculated by directly integrating Eq. (4.39) over the nucleon momentum $\mathbf{p}$ in the nucleus. (Method 1.)

Alternatively, by following Eq. (3.47), the calculation can be carried out as a sum of three $\mathbf{p}$ integrals of $F(\boldsymbol{p},q)$: $F(\boldsymbol{p},q)\cdot(s-u)$, and $F(\boldsymbol{p},q)\cdot(s-u)^2$. The coefficients of the integrals depend on t, as $t = q^2$ is independent of $\mathbf{p}$, but s and u are not. (Method 2.)

The calculation can be also carried out differently, in terms of seven integrals with $F(\boldsymbol{p},q)$. According to Eq. (3.6), $T^{\mu\nu}$ can be expanded in the general tensor form with the five T-coefficients, T_1, T_2, etc., which depend on t. This expansion provides the seven terms to be integrated with $F(\boldsymbol{p},q)$. (Method 3.) This method is used by Smith and Moniz [2] in their treatment of the relativistic Fermi gas model. In the next subsection, we will elaborate on their treatment, also listing the seven integrals.

The three methods are clearly equivalent, as one can see from the way they are formulated. When the on-the-mass-shell kinematics are used to the nucleons in the nucleus, they surely yield the identical result for a given $F(\boldsymbol{p},q)$. When off-the-mass shell kinematics are applied, however, the three methods could end up with yielding different results, depending on which quantities are assumed to go off-shell in each method.

Consider the use of the (effective) nucleon binding energy, $\epsilon \neq 0$, in the Fermi gas model. If the fourth component of q is shifted by ϵ everywhere in the integrand of Eq. (4.39), the three method gives the same result. But, for example, in the third model, if one shifts q in $T^{\mu\nu}$ but leaves q in the five T-coefficients unchanged, the third model yields a different result. One would be motivated to do this by viewing the T-coefficients to be the νN interaction verteces that are immune to the nuclear environment. This is the procedure used in the Smith-Moniz paper without an explicit warning.

Whether this procedure is correct is a dynamical question, involving the heart of nuclear many-body theory. No clear-cut recipe is known for this question and for how to go off-shell in general. Among other various issues, the problem stems from the fact that the use of a single parameter ϵ itself is a crude approximation to the complicated many-body nuclear dynamics. The approximation has a limitation in its application. For example, the use of ϵ is known to fail in the case of electron-nucleus scattering when the cross section is separated into the transverse and longitudinal parts. Note that the uncertainty associated with the off-shell effects is not expected to be always negligible. (For example, see H. W. L. Naus et al., Nucl. Phys. **A509**, 717 (1990) [3]).

G. The relation to direct use of the νN cross section

In the above $\mathbf{p}$ integral, we could use $d\sigma/dt$ instead of $G^2\eta_{\mu\nu}T^{\mu\nu}/2$, which would correspond to a more intuitive description of the reaction process. In fact, some of the Monte carlo codes are based on this description, which are currently used for the simulation and analyses of neutrino experiments. The use of $d\sigma/dt$ makes, however, the expression of the integrand quite complicated and requires much care when it is formulated.

The complication arises simply because the neutrino reaction for $d\sigma/dt$ is on a moving nucleon (with $\mathbf{p} \neq \mathbf{0}$) in the nucleus. For treating this properly, let us go back to the algebra of Subsection III.D and repeat it for the moving nucleon target. Consider the nucleon undergoes the neutrino reaction with its four-momentum changing from $p_1 = (E, \mathbf{p})$ to $(E', \mathbf{p}')$. For now, we assume the nucleon to follow the on-shell kinematics, $E = \sqrt{\mathbf{p}^2 + M^2} \equiv p^0$ and $E' = \sqrt{\mathbf{p}'^2 + M^2}$. The cross section is expressed as

$$
\begin{aligned}
d\sigma &= \frac{(2\pi)^4}{4|k_1 \cdot p_1|} \frac{d^3\boldsymbol{k}_\ell}{2E_\ell(2\pi)^3} \frac{1}{2E'(2\pi)^3} \overline{\sum_i} \sum_f |\mathcal{M}|^2 \delta(E + E_\nu - E' - E_\ell) \\
&= \frac{1}{64\pi^2} \frac{1}{4|k_1 \cdot p_1|} \frac{1}{E_\ell E'} \frac{\pi k_\ell}{E_\nu} dt dk_\ell \frac{\delta(k_\ell - ...)}{|\partial(E + E_\nu - E' - E_\ell)/\partial k_\ell|_t} \overline{\sum_i} \sum_f |\mathcal{M}|^2.
\end{aligned}
\tag{4.41}
$$

Here,

$$
|k_1 \cdot p_1| = E_\nu (E - p\cos\theta) \tag{4.42}
$$

by taking the neutrino incident direction to be along the z axis, and $p \equiv |\mathbf{p}|$. The algebra of the Jacobian become somewhat complicated when $k_\ell = |\mathbf{k}_\ell|$ is treated an independent variable:

$$
|\partial(E + E_\nu - E' - E_\ell)/\partial k_\ell|_t = \left| \frac{1}{E'} \frac{\partial}{\partial k_\ell} (p k_\ell \cos\Omega + E_\nu k_\ell \cos\tau) + \frac{k_\ell}{E_\ell} \right|, \tag{4.43}
$$

where Ω is the angle between $\mathbf{p}$ and $\mathbf{k}_\ell$, satisfying

$$
\cos\Omega = \sin\theta\cos\phi\sin\tau + \cos\theta\cos\tau \tag{4.44}
$$

for $\mathbf{p} = (p, \theta, \phi)$ and $\mathbf{k}_\ell = (k_\ell, \tau, 0)$ in the spherical coordinate system. Noticing that Ω and τ depend on k_ℓ, we obtain after some algebra

$$
|\partial(E + E_\nu - E' - E_\ell)/\partial k_\ell|_t = \frac{E}{E'} \cdot \beta_\ell \cdot (1 - \beta\cos\theta) \cdot \xi^{-1}(\beta, \beta_\ell, \theta, \phi, \tau), \tag{4.45}
$$

where

$$\xi(\beta,\beta_\ell,\theta,\phi,\tau)=\left|\left(1-\frac{\beta}{\beta_\ell}\cdot\frac{1-\beta_\ell\cos\tau}{1-\beta\cos\theta}\cdot\frac{\sin\theta\cos\phi}{\sin\tau}\right)^{-1}\right| \tag{4.46}$$

with $\beta\equiv p/E$ and $\beta_\ell\equiv k_\ell/E_\ell$.

We thus have

$$\begin{aligned}\frac{d\sigma}{dt}(\mathbf{p},q;E_\nu)&=\frac{1}{64\pi E^2(E_\nu-p\cos\theta)^2}\overline{\sum_i}\sum_f|\mathcal{M}|^2\cdot\xi(\beta,\beta_\ell,\theta,\phi,\tau)\\&=\left[\frac{d\sigma}{dt}(\mathbf{p}=0,q;\tilde{E}_\nu)\right]\cdot\frac{|\mathbf{p}|^2}{E^2}\xi(\beta,\beta_\ell,\theta,\phi,\tau).\end{aligned} \tag{4.47}$$

Here,

$$\left[\frac{d\sigma}{dt}(\mathbf{p}=0,q;\tilde{E}_\nu)\right]=\frac{1}{64\pi M^2\tilde{E}_\nu^2}\overline{\sum_i}\sum_f|\mathcal{M}|^2 \tag{4.48}$$

is the neutrino-cross section for the nucleon at rest for the neutrino energy $\tilde{E}_\nu=E_\nu-p\cos\theta$, shifted from E_ν as $\mathbf{p}\to\mathbf{0}$.

$\frac{|\mathbf{p}|^2}{E^2}\dot{\xi}$ is the factorial change for $d\sigma/dt$ as the coordinate system is Lorentz-boosted by $\mathbf{p}\to\mathbf{0}$. Note that s,t, and u are Lorentz invariant and so is $\overline{\sum_i}\sum_f|\mathcal{M}|^2$ The point here is simply that $d\sigma/dt$ is not a Lorentz-invariant quantity in general. Note, as passing, that the incident flux factor $4|k_1\cdot p_1|$ is also not a Lorentz-invariant quantity unless $\mathbf{k}_1$ is in parallel or anti-parallel to $\mathbf{p}_1$.

The factorial change affects the integrated cross section by small amount, but they it affects the angular dependence of the cross section especially appreciably in low-energy kinematical regions. It is indeed *not* suffice to evaluate the known expression of $d\sigma/dt$ for the nucleon at rest ($\mathbf{p}=0$) by Lorentz-transforming E_ν to $\tilde{E}_\nu$, as it is done in some of Monte Carlo codes currently in use. The changes associated with the Lorentz transformation, in the incident neutrino flux and in the $\partial(E+...)/\partial k_\ell$, must also be accounted for.

Actually, when one takes account of all these, one ends up with evaluating the Lorentz-invariant quantity, that is, $G^2\eta_{\mu\nu}T^{\mu\nu}/2$, the value of which remains the same under a Lorentz transformation. It is the quantity that we proposed to evaluate using Method 1, as described in the previous subsection. We believe the use of $G^2\eta_{\mu\nu}T^{\mu\nu}/2$ instead of $d\sigma/dt$ provides the most convenient starting point for more refined Monte Carlo codes by using a more realistic $F(\mathbf{p},q)$ than that of the Fermi gas mode, also including effects of the final-state interactions.

H. W and T coefficients, and factorized nuclear cross section

The W coefficients are now expressed as linear combinations of the T coefficients as

$$W_1=a_1T_1+\frac{a_2-a_3}{2}M^2T_2 \tag{4.49}$$

$$W_2=M^2\left[a_4-\frac{2q^0}{q}a_5+\frac{(q^0)^2}{q^2}a_3-\frac{(q^0)^2-q^2}{2q^2}(a_2-a_3)\right]T_2 \tag{4.50}$$

$$W_\alpha = T_\alpha a_1 + \frac{2M}{q} a_6 T_\beta + \frac{M^2}{2q^2}(3a_3 - a_2)T_2 \tag{4.51}$$

$$W_\beta = M\left(a_7 - \frac{q^0}{q}a_6\right)T_\beta + \frac{M^2}{q^2}\left[\frac{q^0}{2}(a_2 - 3a_3) + qa_5\right]T_2 \tag{4.52}$$

$$W_8 = M\left(a_7 - \frac{q^0}{q}a_6\right)T_8, \tag{4.53}$$

following the work of Smith and Moniz [2]. Here, a_i's depend on q and expressed as

$$a_1 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q) \tag{4.54}$$

$$a_2 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{p^2}{M^2} \tag{4.55}$$

$$a_3 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{p^2\cos^2\tau}{M^2} \tag{4.56}$$

$$a_4 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{E_p^2}{M^2} \tag{4.57}$$

$$a_5 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{E_p p\cos\tau}{M^2} \tag{4.58}$$

$$a_6 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{p\cos\tau}{M} \tag{4.59}$$

$$a_7 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q)\frac{E_p}{M}, \tag{4.60}$$

where we denote $p \equiv |\boldsymbol{p}|$, $\cos\tau = \boldsymbol{p}\cdot\boldsymbol{q}/|\boldsymbol{p}\cdot\boldsymbol{q}|$, and $E_p = \sqrt{\boldsymbol{p}^2 + M^2}$. Note that the T coefficients are previously expressed in terms of the nucleon form factors.

When we neglect the nucleon momentum $|\boldsymbol{p}|$ with respect to the nucleon mass M, we have

$$a_1 = a_4 = a_7 = \int d^3\boldsymbol{p} F(\boldsymbol{p}, q) \equiv a \tag{4.61}$$

and all other a_i's vanish. Each W_i then becomes proportional to each T_i as

$$W_1 = aT_1 \tag{4.62}$$

$$W_2 = aM^2T_2 \tag{4.63}$$

$$W_\alpha = aT_\alpha \tag{4.64}$$

$$W_\beta = aMT_\beta \tag{4.65}$$

$$W_8 = aMT_8. \tag{4.66}$$

The neutrino-nucleus differential cross section is then factorized to the two parts, the neutrino-nucleon scattering and the other, a, the dynamical response of the target nucleus:

$$\begin{aligned}\frac{d\sigma}{dE_\ell d\Omega} = \frac{G^2}{16\pi^2}\frac{k_\ell E_\ell}{M_T}\Bigg\{&\left(1 - \frac{k_\ell}{E_\ell}\cos\theta\right)(2W_1 + m^2 W_\alpha) + \left(1 + \frac{k_\ell}{E_\ell}\cos\theta\right)W_2 \\ &- \frac{2m^2}{E_\ell}(W_\beta + W_8) + 2(E_\nu + E_\ell)\left(1 - \frac{k_\ell}{E_\ell}\cos\theta\right)W_8\Bigg\}\end{aligned} \tag{4.67}$$

$$
\begin{aligned}
&= \frac{M^2}{M_T} \cdot a \cdot \left\{ \left[1 + \frac{E_\nu}{M} \left(1 - \frac{E_\ell}{k_\ell} \cos\theta \right) \right] \frac{d\sigma}{d\Omega} \right\} \\
&= \frac{M^2}{M_T} \cdot a \cdot \left(\frac{d\sigma}{d\Omega} \right)_{NRF} .
\end{aligned} \tag{4.68}
$$

where $(d\sigma/d\Omega)_{NRF}$ is the neutrino-nucleon cross section with no recoil factor.

These expressions are generally valid with the assumptions specified above.

V. INCLUSIVE NEUTRINO-NUCLEUS QUASI-ELASTIC SCATTERING; FERMI-GAS MODEL AND BEYOND

A. The relativistic Fermi gas model

We now assume that the initial nuclei is described by a (zero-temperature) Fermi gas obeying on-shell relativistic kinematics. All nucleon states are occupied by the nucleons with the momenta smaller than the effective Fermi momentum, p_F, and

$$
\hat{H} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p} E_p a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}. \tag{5.1}
$$

Note that we denote $p = |\mathbf{p}|$. The spectral functions,

$$
\begin{aligned}
P_h(\boldsymbol{p}, \omega) &= \frac{1}{2P_A^0} \langle P_A, i | a^\dagger_{\boldsymbol{p}} \delta(\hat{H} - P_A^0 - \omega) a_{\boldsymbol{p}} | P_A, i \rangle \\
P_p(\boldsymbol{p}, \omega) &= \frac{1}{2P_A^0} \langle P_A, i | a_{\boldsymbol{p}} \delta(\hat{H} - P_A^0 - \omega) a^\dagger_{\boldsymbol{p}} | P_A, i \rangle,
\end{aligned} \tag{5.2}
$$

becomes

$$
\begin{aligned}
P_h^F(\boldsymbol{p}, \omega) &= \frac{1}{2P_A^0} \delta(P_A^0 - p^0 - P_A^0 - \omega) \langle P_A, i | a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} | P_A, i \rangle \\
&= \delta(p^0 + \omega) V \theta(p_F - p) \\
P_p^F(\boldsymbol{p}, \omega) &= \frac{1}{2P_A^0} \delta(P_A^0 + p^0 - P_A^0 - \omega) \langle P_A, i | a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} | P_A, i \rangle, \\
&= \delta(p^0 - \omega) V \theta(p - p_F).
\end{aligned} \tag{5.3}
$$

$F(\boldsymbol{p}, q)$ is then

$$
\begin{aligned}
F(\boldsymbol{p}, q) &= \frac{M_T}{(2\pi)^3 V} \int \frac{d\omega'}{p^0(p^0 + \omega)} P_h(\boldsymbol{p}, \omega') P_p(\boldsymbol{p} + \boldsymbol{q}, \omega - \omega') \\
&= \frac{V}{(2\pi)^3} \frac{M_T}{E_p(E_p + \omega)} \delta(E_{p+q} - E_p - \omega) \theta(p_F - |\boldsymbol{p}|) \theta(|\boldsymbol{p} + \boldsymbol{q}| - p_F).
\end{aligned} \tag{5.4}
$$

where $E_p \equiv p^0 = \sqrt{\mathbf{p}^2 + M^2}$ and $E_{p+q} = \sqrt{(\mathbf{p} + \mathbf{q})^2 + M^2}$ for the on-shell kinematics. The spacial normalization volume V is

$$
V = 6\pi^2 \cdot N / p_F^3, \tag{5.5}
$$

computed from the phase-space quantisation condition

$$V \cdot \frac{4\pi}{3} p_F^3/(2\pi)^3 = N, \tag{5.6}$$

where N is the neutron number and is replaced by the atomic number Z for the $(\bar{\nu}, \ell^+)$ reaction. Using the above $F(\boldsymbol{p}, q)$ in Eqs. (4.39) and (4.40) with Eqs. (3.48) – (3.50), one calculate $d^2\sigma/dE_\ell d\Omega$ numerically. (Method 1 in Subsection IV. F.) Alternatively, applying the expressions in Subsection IV. H, one can calculate a_i's analytically, then W_i's, and thus the crosss section. The analytical forms of a_i's and W_i's are given in the paper by Smith and Moniz [2]. (Method 3 in Subsection IV. F.) As noted in Subsection IV. F, the two method yield the same result for the on-shell kinematics.

When all nucleons in the nucleus are assumed to be bound by the effective binding energy ϵ common to all nucleons, the nucleon energies are changed as

$$\begin{aligned} E_p &\to \sqrt{\boldsymbol{p}^2 + M^2} - \epsilon \\ E_{p+q} &\to \sqrt{(\boldsymbol{p}+\boldsymbol{q})^2 + M^2} - \epsilon' \end{aligned} \tag{5.7}$$

where $\epsilon' \approx 0$ for $|\mathbf{q}|$ close to p_F. Through the energy delta function in Eq. (5.4), ω increased by $\epsilon - \epsilon'$ is now needed to induce the same reaction as that without the effective binding energies.

The above $F(\boldsymbol{p}, q)$ yields the nonrelativistic limit of the a_i coefficients

$$\begin{aligned} a &= \frac{M_T}{M^2} \frac{V \cdot 4\pi^3 p_F/3}{(2\pi)^3} \cdot \frac{1}{4\pi^3 p_F^3/3} \int \frac{d^3 p M^2}{p^0(p^0+\omega)} \theta(p_F - |\boldsymbol{p}|)\theta(|\boldsymbol{p} - q| - p_F)\delta(p'^0 - p^0 - \omega) \\ &= N \cdot \frac{M_T}{M^2} R(\boldsymbol{q}, \omega), \end{aligned} \tag{5.8}$$

where $R(\boldsymbol{q}, \omega)$ is the R-factor defined in T. K. Gaisser and J. S. O'Connell, Phys. Rev. **D34**, 822 (1986) [4]. We thus obtain in the nonrelativistic limit

$$\frac{d\sigma}{dE_\ell d\Omega} = N R(\boldsymbol{q}, \omega) \cdot \left(\frac{d\sigma}{d\Omega}\right)_{NRF} \tag{5.9}$$

in agreement with Gaisser-O'Connell's expression. Note, however, their $d\sigma/d\Omega$ expression involves non-relativistic approximations.

Now, the reason is quite clear in this Fermi gas model why no recoil factor appears in the nuclear reaction cross section. The delta function describing the energy conservation in the above $F(\boldsymbol{p}, q)$ disappears upon $\boldsymbol{p}$ integration in a_i's, and can thus generate no recoil factor. That is, the energy conservation is applied on the overall neutrino-nucleus reaction process. In fact, Gaisser and O'Connell shows that in the limit of $p_F \to 0$, we obtain

$$R(\boldsymbol{q}, \omega) \to \delta(E_\ell - ...)/(\text{recoil factor}), \tag{5.10}$$

so that the above $d\sigma/dE_\ell d\Omega$ becomes N times $d\sigma/d\Omega$ for the neutron after the E_ℓ integration.

In the following two figures, we illustrate the parameter dependence of the effective neutrino-nucleon cross sections in the Fermi-gas model. In the subsequent two figures, we also show the consequence on the cross sections due to the factorization approximation.

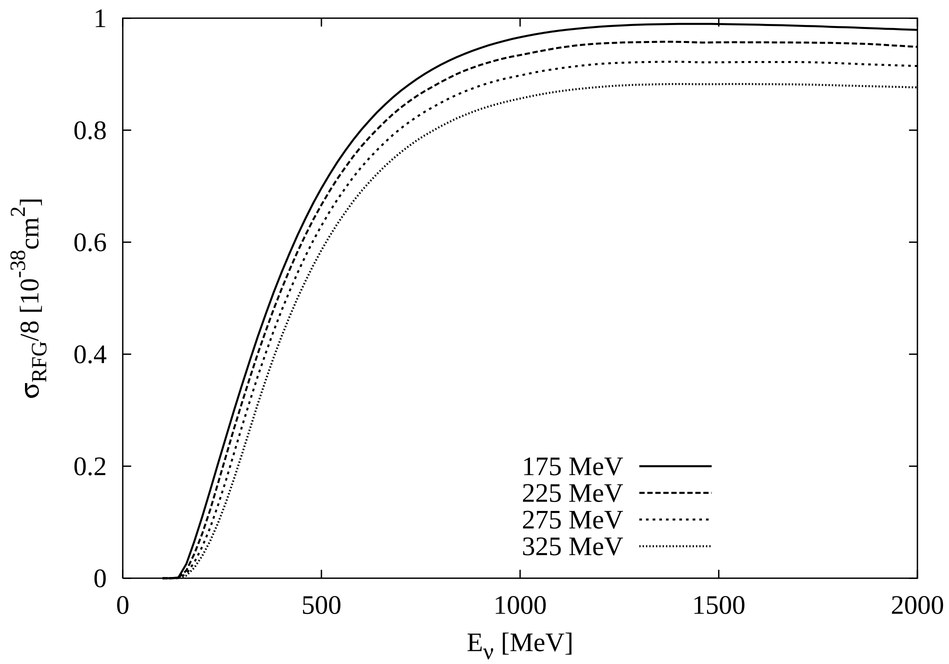

FIG. 7. The cross sections of μ-neutrino and bound nucleon scattering in realtivistic Fermi gas model for various Fermi momenta. Binding energy is fixed to 27 MeV.

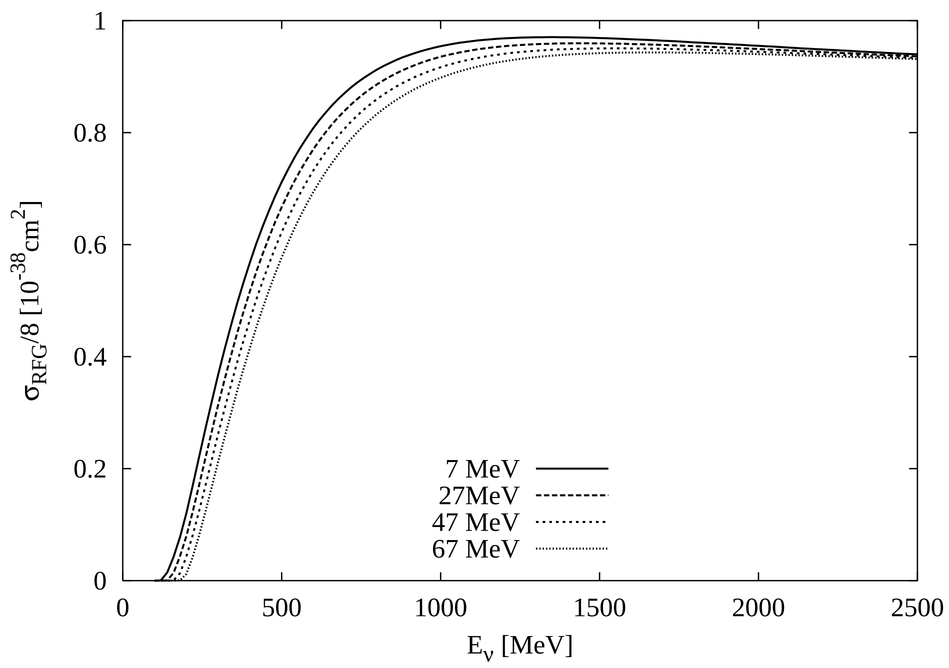

FIG. 8. The cross sections of μ-neutrino and bound nucleon scattering in realtivistic Fermi gas model for various values of binding energy. Fermi momentum is fixed to 225 MeV.

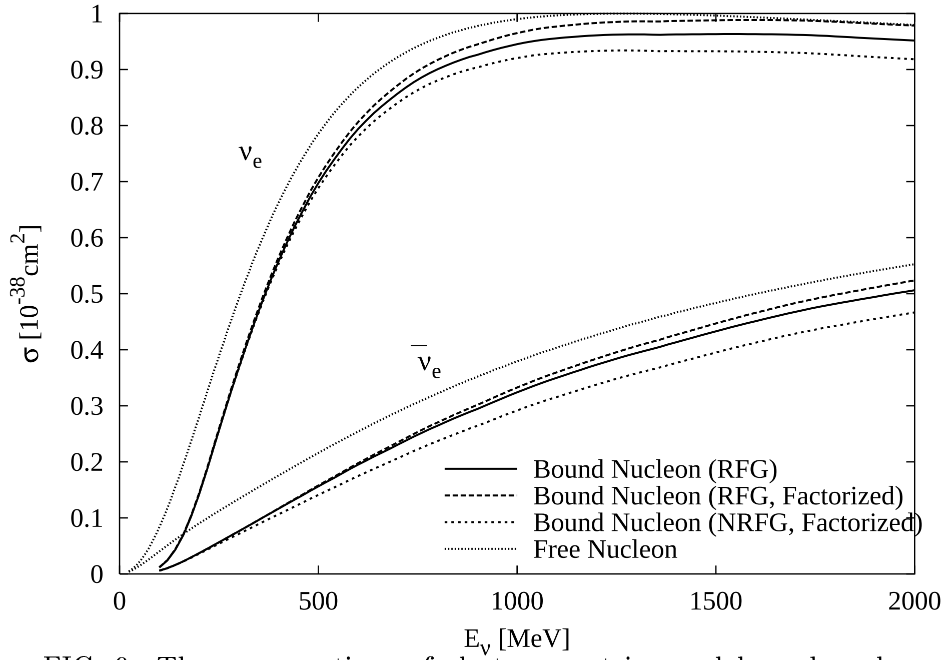

FIG. 9. The cross sections of electron-neutrino and bound nucleon scattering. Each curve corresponds to relativistic Fermi gas model, factorized relativistic Fermi gas and factorized non-relativistic Fermi gas model. The free nucleon case is also drawn for comparison. Fermi momentum and binding energy are 225 MeV and 27 MeV for all models.

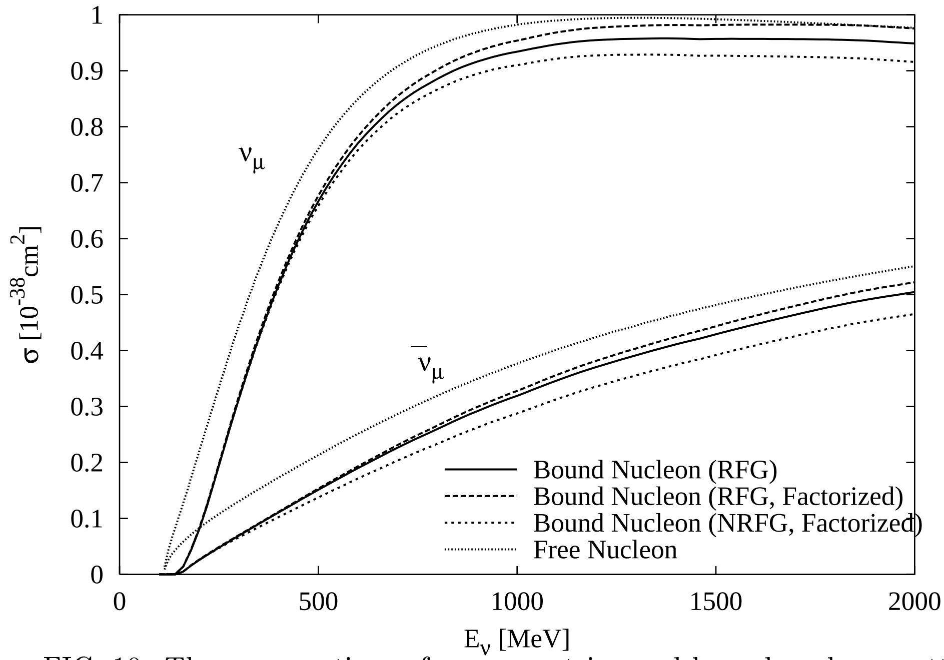

FIG. 10. The cross sections of muon-neutrino and bound nucleon scattering. See the previous figure for explanation of the curves.

B. Relativistic Fermi gas model with single-particle potential

As we noted in Subsection IV. F, the use of the effective binding energy ϵ in the Fermi-gas model is an approximation for complicated nuclear many-body dynamics by a constant (plus another constant, p_F, usually.) The approximation is known to be inadequate for describing the transverse and longitudinal inclusive cross sections of elctron scattering. The use of the approximation is therefore expected to be especially problematic when one tries to *predict* the neutrino cross section using the parameter values phenomenologically extracted from the electron scattering data for a similar kinematics. The electron and neutrino cross sections differ in general in their transverse and longitudinal compositions.

One may expect an improvement by taking account of a piece of physics that the energy of the nucleon in nuclei actually depends on its momentum $\mathbf{p}$. E_p is now a function of $\mathbf{p}$ beyond the on-shell relativistic kinematics,

$$E_p(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + M^2} + V(\boldsymbol{p}). \tag{5.11}$$

The function may also depend on p_F,

$$E_p(\boldsymbol{p}; p_F) = \sqrt{\boldsymbol{p}^2 + M^2} + V(\boldsymbol{p}; p_F). \tag{5.12}$$

Furthermore, when one follows many-body dynamics closely, one expects the relation between E_p and $\mathbf{p}$ to be generally nonlinear,

$$E_p(\boldsymbol{p}; p_F) = \sqrt{\boldsymbol{p}^2 + M^2} + V(\boldsymbol{p}, E_p; p_F), \tag{5.13}$$

which could represent a fully complicated dispersion relation of E_p and $\mathbf{p}$ in nuclei.

As a concrete example, let us consider Eq. (5.12). One can view $V(\boldsymbol{p}; p_F)$ to be (the real part of) an average nucleon-nucleus (optical) potential in nuclear matter at the effective Fermi momentum p_F. When p_F is taken as a constant parameter for the entire nucleus (rather than a quantity depending on the local density), Eq. (5.12) tells that this is an extension of the effective binding energy corresponding to

$$\epsilon \to -V(\boldsymbol{p}; p_F). \tag{5.14}$$

$V(\boldsymbol{p}; p_F)$ has been calculated in various ways in different degrees of sophistication. The following two figures show two examples of the calculations by Brieva and Rook [5] and by Benhar et al. [6] appeared in 1977 and in 1992, respectively. The Brieva-Rook calculation is one of the old calculations and uses a self-consistent single-particle energy spectrum with Hamada-Johnston nucleon-nucleon potential. The Benhar et al. calculation is perhaps the most sophisticated one by carrying out a complicated many-body calculation based on orthogonal correlated basis functions of the nuclear matter using the realistic Urbana v_{14} nucleon-nucleon potential with three-nucleon interactions. In spite of the two different approaches, we see the two figures give approximately the same potential at the nuclear matter density ($p_F \approx 270$ MeV/c). Though not explicitly shown, the potentials would be similar also at the effective p_F of a little greater than 200 MeV. Furthermore, Brieva and Dellafiore showed that the use of the Brieva-Rook potential for $p_F = 217$ MeV/c in the

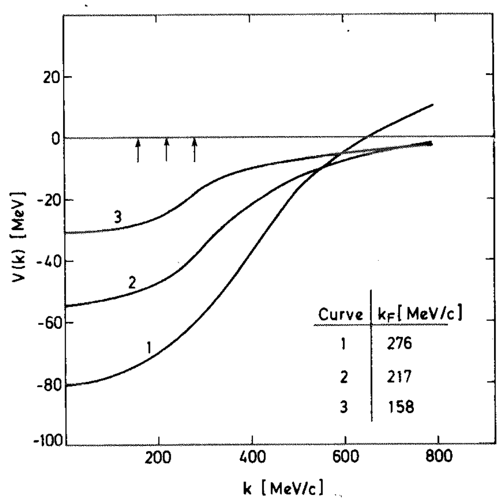

FIG. 11. Real part of the nucleon-nucleus optical potential obtained from Hamada-Johnston internucleon force as a function of the nucleon momentum. The arrows mark the value of k_F appropriate for the different curves. Adopted from F.A. Brieva and A. Dellafiore, Nucl. Phys. A292, 445(1977).

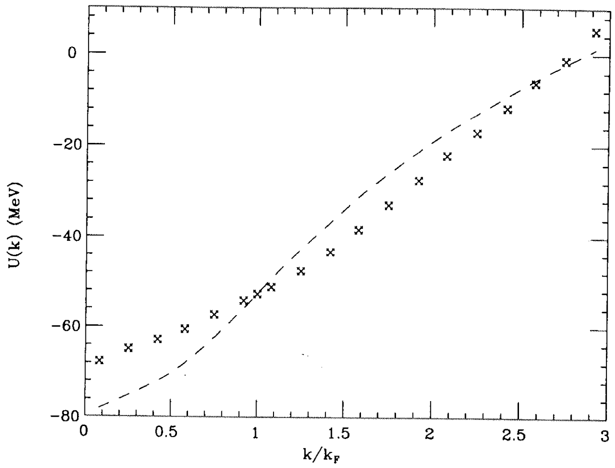

FIG. 12. Real part of the optical potential in nuclear matter. The dashed line are the variational estimates; the crosses are the perturbative results. Adopted from O. Benhar *et al.*, Nucl. Phys. A550, 201(1992).

Fermi-gas model reasonably successfully describes the quasi-elastic cross sections from ^{12}C as a function of the energy transfer ω for the 500 MeV incident electron at 60^o.

Because of this success, can we safely apply it to our neutrino calculations? The answer is no, in general, as we do not expect that the limitation of the two-constant parameter approximation noted in the beginning of this subsection can be resolved by introducing the momentum dependence in the potential. Furthermore, the use of such approximation requires much care, for not exceeding its limitation. For example, if one should try to extract some information of the ejected proton within this PWIA, one would end up with unrealistic results. We will discuss this point in the next subsection.

C. Use of Spectral functions

The approximation described in the preceding subsection is to shift p^0 in the spectral functions for the Fermi-gas model,

$$P_h^F(\boldsymbol{p},\omega) = \delta(p^0+\omega)V\theta(p_F-p)$$
$$P_p^F(\boldsymbol{p},\omega) = \delta(p^0-\omega)V\theta(p-p_F), \tag{5.15}$$

by the amount of $-V(\mathbf{p};p_F)$. V arises as a consequence of the mutual interactions of the nucleons in nuclei. As such interactions are not considered in the Fermi-gas model, the inclusion of this aspect of physics is an improvement in principle. The improvement brings up, however, a problem of what to do with $\theta(p_F-p)$. Once we allow the nucleons to interact each other, the momentum distribution of the nucleons, $n(p)$, deviates from $\theta(p_F-p)$. (This point is also applicable for the use of ϵ, as the nucleon binding is generated only by the nucleon interactions.)

To be consistent, one should use $n(p)$ that is obtained by the same procedure to yield V itself. As this is difficult to do, one may apply $n(p)$ extracted from experiments. Though it may appear to be a good pragmatic compromise, we must be aware that the V's shown in Figs. 11 and 12 are for nuclear matter of various densities, not for finite nuclei.

In order to see why the distinction is a serious issue, we directly compare the hole spectral functions P_h's for nuclear matter and for a finite nucleus. Figures 13 and 14 are of works of O. Benhar and his collaborators [7] by detailed many-body calculations based on correlated basis functions. In the figures, E is the removal energy (to remove a nucleon from the nucleus), corresponding to $-\omega$. The nuclear matter density of 3/4 of the full density ρ_0 roughly corresponds to the effective Fermi momentum of $p_F = 217$ MeV that has appeared above.

First of all, the two figures are functions of both of the energy and momentum, implying the nucleon interactions do not provide a unique (nonlinear or otherwise) dispersion relation of the two quantities, such as Eq. (5.11) - (5.13). That is, for a fixed energy value, we have finite probabilities of the nucleon having various momenta, even to quite high momenta. The high momentum distributions are resulted from the strong nucleon interactions at short distances and are related to strong nuclear correlations at short distance.

Figure 13 shows that the plateau follows roughly the corresponding curve in Fig. 11. We can regard the curve in Fig. 11 (and thus the corresponding one in Fig. 12, though not shown) to be a crude approximation to the spectral function of Fig. 13. How crude the approximation is a different issue, however.

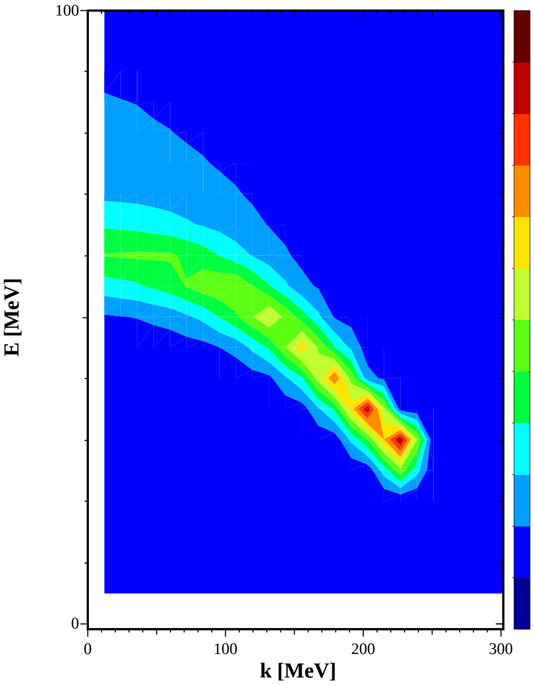

FIG. 13. Spectral function of nuclear matter whose density is 3/4 ρ_0.

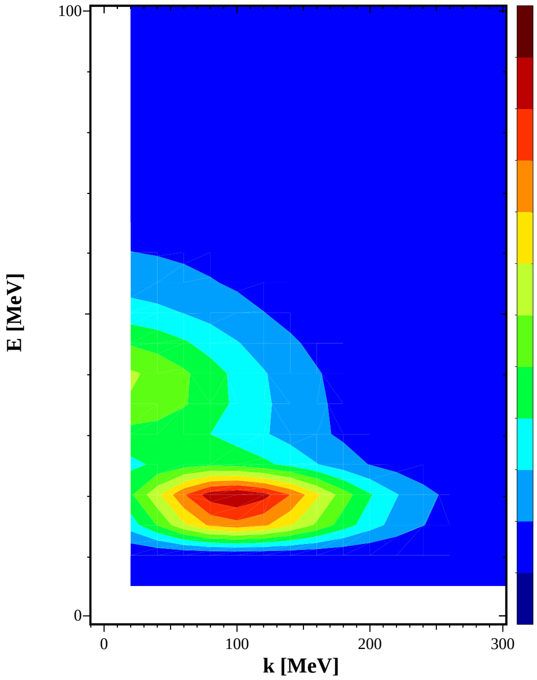

FIG. 14. Spectral function of ^{12}C.

Figure 13 and Fig. 14 are, however, quite different. Figure 14 shows two groups of densed distributions. They correspond to the two major shells in ^{12}C, as their removal energies are about 20 and 40 MeV. Grouping in the distribution of the spectral function is, of course, a natural quantum-mechanical consequence of a finite-size system, in contrast to an infinite-size system such as nuclear matter.

The most important aspect in the difference between the two figures are that the spectral function is distributed more in the reguin of high momentum and large energy in Fig. 13. That is, Fig. 13 or its crude approximation Fig. 11, implies that the nucleons are more tightly bound with larger momenta than what Fig. 14 shows. This is the reason why the use of $-V(\mathbf{p};p_F)$ would give erroneous results for the information of the ejected nucleons, say, in the PWI approximation.

VI. FINAL-STATE INTERACTIONS AND SEMI-INCLUSIVE REACTIONS

REFERENCES

[1] F. Boehm and P. Vogel, *'Physics of Massive Neutrino'*, Cambridge University Press (1992).
[2] R.A. Smith and E.J. Moniz, Nucl. Phys. **B43**, 605(1972).
[3] H.W.L. Naus *et al.*, Nucl. Phys. **A509**, 717 (1990).
[4] T.K. Gaisser and J.S. O'Connell, Phys. Rev. **D34**, 822 (1986).
[5] Figure 1 in F. A. Brieva and A. Dellafiore, Nucl. Phys. **A292**, 445 (1977).
[6] O. Benhar, A. Fabrocini, and S. Fantoni, Nucl. Phys. **A550**, 201 (1992).
[7] Private communication from O. Benhar. The methods of the calculations are described in O. Benhar, A. Fabrocini, S. Fantoni, and I. Sick, Nucl. Phys. **A579**, 493 (1994) and references therein.