Video Meeting
NOV. 30, 1999

SciFi Track Fitting with the Kalman Filter

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INTRODUCTION

- SciFi geometrical limit of the measurement error: $0.7\text{mm}/\sqrt{12}$ =202 μm

- Standard Global Fit : ≈1.5mm of residual

- Alignment Error : ≈0.5mm (?) → ≈540μm only.

- Multiple Coulomb scattering (MCS) effect for a muon with p=1GeV, L=9cm, and $X_R = 33.6$cm (Al+Water):

$$\sigma_X(\sigma_Y) = \frac{0.0141E}{p^2} L \sqrt{\frac{L}{3X_R}} = 38\mu m.$$

- → not a small effect and Kalman Filtering can account for the MCS.

- Originally introduced by the engineer Kalman. P.Billoir and R. Fruhwirth are pioneers of the algorithm in HEP.

- The algorithm is recursive and stepwise → additive.
  Equivalent to Standard global fit with the MCS.
  Use analytic equation → Calculation is fast. But, needs initial guess.

KALMAN FILTERING

- System Equation.

$$x_{k,t} = F_{k-1}x_{k-1,t} + w_{k-1}, \tag{1}$$

$x_{k,t}$: State Vector (Slopes and Intercepts in SciFi)
F_{k-1} : Evolution matrix
w_{k-1} : A random disturbance (MCS in SciFi)

- Measurement $\approx$ state vector.

$$m_k = H_k x_{k,t} + \epsilon_k. \tag{2}$$

m_k : Measurement points (x or y in SciFi)
H_k : Measurement projection matrix
ϵ_k : Intrinsic Resolution (Meas. + Alignment)

- Kalman Filtering Prescriptions For Fitting

Assumptions

$$\begin{aligned} E\{w_k\} = 0, cov\{w_k\} = Q_k \\ E\{\epsilon_k\} = 0, cov\{epsilon_k\} = V_k = G_k^{-1}. \end{aligned} \tag{3}$$

Frequently used Terms. ;
Prediction, *Filtering*, and *Smoothing*

Prediction: State vector estimation @ a future time.
$x_k^{k-1} = F_{k-1}x_{k-1}$: Estimated state vector
$C_k^{k-1} = F_{k-1}C_{k-1}F_{k-1}^T + Q_{k-1}$: Covariance of x_k^{k-1}
Q_{k-1} :Covariance matrix of w_k.

Filtering: State vector estimation @ present
$x_k = x_k^{k-1} + K_k(m_k - H_k x_k^{k-1})$: Estimated state vector
$C_k = (I - K_k H_k)C_k^{k-1}$: Covariance of x_k
$K_k = C_k^{k-1}H_k^T(V_k + H_k C_k^{k-1}H_k^T)^{-1}$: Kalman gain mat
V_k : Covariance matrix of the ϵ_k, OR
$x_k = C_k[(C_k^{k-1})^{-1}x_k^{k-1} + H_k^T(V_k)^{-1}m_k]$
$C_k = [(C_k^{k-1})^{-1}x_k^{k-1} + H_k^T(V_k)^{-1}H_k]^{-1}$

Smoothing: Improvement in estimation of a state vector
(*Prediction* + *Filtering*)
$x_k^n = x_k + A_k(x_{k+1}^n - x_{k+1}^k)$: Smoothed state vector
$C_k^n = C_k + A_k(C_{k+1}^n - C_{k+1}^k)A_k^T$: Covariance of x_k^n
$A_k = C_k F_k^T(C_k^k + 1)^{-1}$: Smoother gain matrix
Note:$C_k F_k^T = F^{-1}(C_{k+1}^k - Q_k)$

• Residual and Its Covariance

Filtering

$r_k = m_k - H_k x_k$: Residual

$R_k = (I - H_k K_k) V_k = V_k - H_k C_k H_k^T$: Covariance of r_k

Smoothing

$r_k^n = m_k - H_k x_k^n$: Residual

$R_k^n = V_k - H_k C_k^n H_k^T$: Covariance of r_k^n or

$R_k^n = R_k - H_k A_k (C_{k+1}^n - C_{k+1}^k) A_k^T H_k^T$

• χ^2 and Individual χ^2 : Good and Bad

Filtering : Global χ^2

$\chi^2_{global} = \sum_{k=1}^{nhit} r_k^T (R_k)^{-1} r_k$

Smoothing : Individual χ^2_S

$\chi^2_S = r_k^{nT} (R_k^n)^{-1} r_k^n$

Note: χ^2_S is $pull^2$

• Noise Hit Removal:Inverse Kalman Filtering

To obtain an optimal estimate with a particular hit.

$x_k^{n*} = x_k^n + K_k^{n*}(m_k - H_k x_k^n)$: x_k^n without k-th hit

$K_k^{n*} = C_k^n H_k^T (-V_k + H_k C_k^n H_k^T)^{-1}$: Note '-' sign in V_k

$C_k^{n*} = (I - K_k^{n*} H_k) C_k^n$: Covariance of x_k^{n*}

TRACKING MODEL : IMPLEMENTATION FOR THE SCIFI

• Choice of Tracking Model : Very important

• Track Equation for the SciFi

$$\begin{aligned} w &= x\delta_{x,z} + y\delta_{y,z} \\ &= (\alpha_x(z - z_k) + \beta_x)\delta_{x,z} + (\alpha_y(z - z_k) + \beta_y)\delta_{y,z}, \end{aligned} \quad (4)$$

$\delta_{x,z}$ ($\delta_{y,z}$) : 1 if x-z (y-z) measurement

• Measurement at z_k

$$\begin{aligned} w_k &= x_k\delta_{x,k} + y_k\delta_{y,k} \\ &= \beta_{x,k}\delta_{x,k} + \beta_{y,k}\delta_{y,k}. \end{aligned} \quad (5)$$

• Evolution equations

$$\begin{aligned} \beta_{x,k} = &\ \alpha_{x,(k-1)}[(z_k - z_{(k-1)})\delta_{x,k}\delta_{x,(k-1)} + \\ &\ (z_k - z_{k,(last)})\delta_{x,k}\delta_{y,(k-1)}] \\ &\ +\beta_{x,(k-1)} \\ \beta_{y,k} = &\ \alpha_{y,(k-1)}[(z_k - z_{(k-1)})\delta_{y,k}\delta_{y,(k-1)} + \\ &\ (z_k - z_{k,(last)})\delta_{y,k}\delta_{x,(k-1)}] \\ &\ +\beta_{x,(k-1)} \\ \alpha_{x,k} = &\ \alpha_{x,(k-1)} \\ \alpha_{y,k} = &\ \alpha_{y,(k-1)}, \end{aligned}$$

where

$$
\begin{aligned}
\delta_{x,k} &= 1\; if\; i = X - Z\; measurement \\
&\quad 0\; if\; i = Y - Z\; measurement, \\
\delta_{y,k} &= 1\; if\; i = Y - Z\; measurement \\
&\quad 0\; if\; i = X - Z\; measurement, \\
\delta_{x,k}\delta_{y,k} &= 0, \\
(\delta_{x,k})^2 &= 1,\; and \\
(\delta_{y,k})^2 &= 1.
\end{aligned}
$$

- Evolution Matrix and Measurement Projection Matrix

$F_k = \frac{\partial \vec{x}_k}{\partial \vec{x}_{k-1}}$ and
$H_k = \frac{\partial w_k}{\partial \vec{x}_k}$

Multiple Scattering

- Covariance Matrix :

We use Gaussian approximation with the projected angular distribution of a width ;

$$\theta_0 = \frac{13.6MeV}{\beta cp} z\sqrt{x/X_0}[1 + 0.038ln(x/X0)], \tag{6}$$

Here, p, βc, and z : momentum, velocity, and charge
X_0 : radiation length, x : thickness of the scatterer

Material Assumption :
G10+Honey Comb (Length=2cm, density=0.326g/cm^2)
Aluminium (Length=0.36cm, density=2.7g/cm^2)
Water (Length=5.64cm, density=1.0g/cm^2)
Material length is the thick when normal incidence
Aluminium Length =1.5cm gives good pulls
(Note 25cm barriers) $\rightarrow$ Default in the code

The Covariance Matrix due to the MCS:

$$Q_{k-1} = \theta_0^2 \begin{pmatrix} (1+\alpha_{x,k}^2)\frac{l^2}{3} & \alpha_{x,k}\alpha_{y,k}\frac{l^2}{3} & (1+\alpha_{x,k}^2)\kappa\frac{l}{2} & \alpha_{x,k}\alpha_{y,k}\kappa\frac{l}{2} \\ \alpha_{x,k}\alpha_{y,k}\frac{l^2}{3} & (1+\alpha_{y,k}^2)\frac{l^2}{3} & \alpha_{x,k}\alpha_{y,k}\kappa\frac{l}{2} & (1+\alpha_{y,k}^2)\kappa\frac{l}{2} \\ (1+\alpha_{x,k}^2)\kappa\frac{l}{2} & \alpha_{x,k}\alpha_{y,k}\kappa\frac{l}{2} & (1+\alpha_{x,k}^2)\kappa^2 & \alpha_{x,k}\alpha_{y,k}\kappa^2 \\ \alpha_{x,k}\alpha_{y,k}\kappa\frac{l}{2} & (1+\alpha_{y,k}^2)\kappa\frac{l}{2} & \alpha_{x,k}\alpha_{y,k}\kappa^2 & (1+\alpha_{y,k}^2)\kappa^2 \end{pmatrix},$$

$\kappa = \sqrt{1 + \alpha_{x,k}^2 + \alpha_{y,k}^2}$
l : the thickness of the scatterer.

- Generation of multiple scattering

Very helpful in checking the Kalman filtering

Track scatters at (x_0, y_0, z_0) and θ_0 is generated at the track coordinate system

Track Coordinate System (TCS) : Origin at (x_0, y_0, z_0) and z-axis aligned with the unscattered track.

Transformation between TCS and SciFi : 3×3 rotation matrix (R)

$$a \begin{pmatrix} \alpha_{x,k} \\ \alpha_{y,k} \\ 1 \end{pmatrix} = a' R \begin{pmatrix} tan(\theta_1) \\ tan(\theta_2) \\ 1 \end{pmatrix},$$

a and a' : normalization constants for unit vectors
From above, slopes in the SciFi coordinate system:

$$\alpha_{x,k} = \frac{R_{11} tan\theta_1 + R_{12} tan\theta_2 + \frac{\alpha_{0,x,k}}{\sqrt{1+\alpha^2_{0,x,k}+\alpha^2_{0,y,k}}}}{R_{31} tan\theta_1 + R_{32} tan\theta_2 + \frac{1}{\sqrt{1+\alpha^2_{0,x,k}+\alpha^2_{0,y,k}}}},$$
$$\alpha_{y,k} = \frac{R_{21} tan\theta_1 + R_{22} tan\theta_2 + \frac{\alpha_{0,x,k}}{\sqrt{1+\alpha^2_{0,x,k}+\alpha^2_{0,y,k}}}}{R_{31} tan\theta_1 + R_{32} tan\theta_2 + \frac{1}{\sqrt{1+\alpha^2_{0,x,k}+\alpha^2_{0,y,k}}}}. \qquad (7)$$

The new intercepts in the SciFi coordinate system:

$$\beta_{x,k} = \beta_{x,(k-1)} + (z_k - z_{(k-1)})\alpha_{x,k},$$
$$\beta_{x,k} = \beta_{y,(k-1)} + (z_k - z_{(k-1)})\alpha_{y,k}. \qquad (8)$$

Finally, use a reconstructed track's slopes and intercepts as a seed track to generate a fake track.

Test of Kalman Fitter for the SciFi

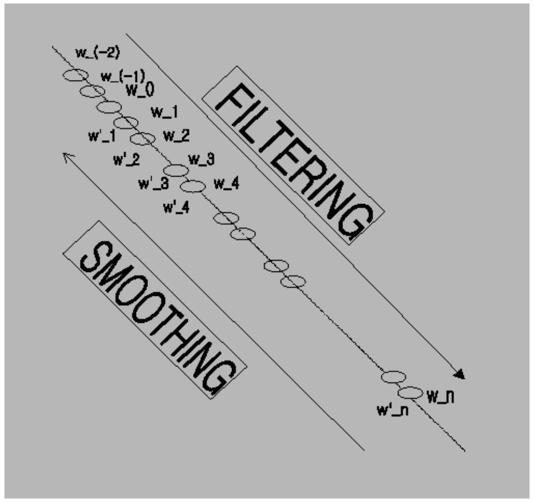

Figure 1: Flow of Kalman filtering. w_1,...,w_n are filtered positions while w'_1,...,w'_n are smoothed positions. $w_{(-2)}$, $w_{(-1)}$, w_0 are positions of initial guess, and two artificial hits. If w_1 is an X-Z hit, w_0 is set to be an artificial Y-Z hit and $w_{(-1)}$ is set to be an artificial X-Z hit, and vice versa.

- Tests on the MC events

First select muon track
Associate hits with MC true positoin
Residuals: $\approx$ overall position resoluton $|pull|$ <3 are removed as described before. Here,

$$p_i = \frac{r_k}{\sqrt{R_k}}, \text{k=1,2,...,nhit} \tag{9}$$

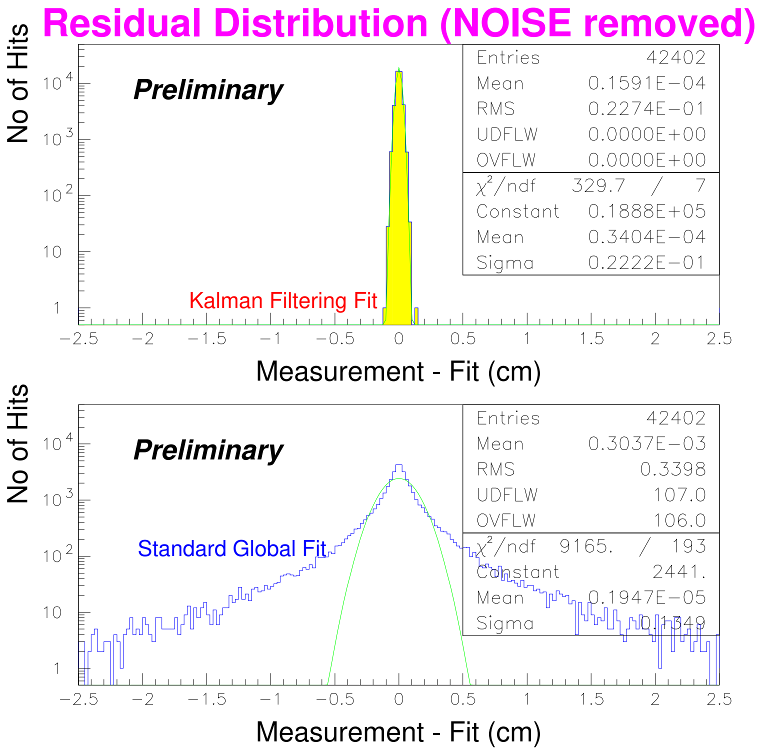

Figure 2: A comparison of residual distributions between the Kalman filtering fit and the standard global fit.

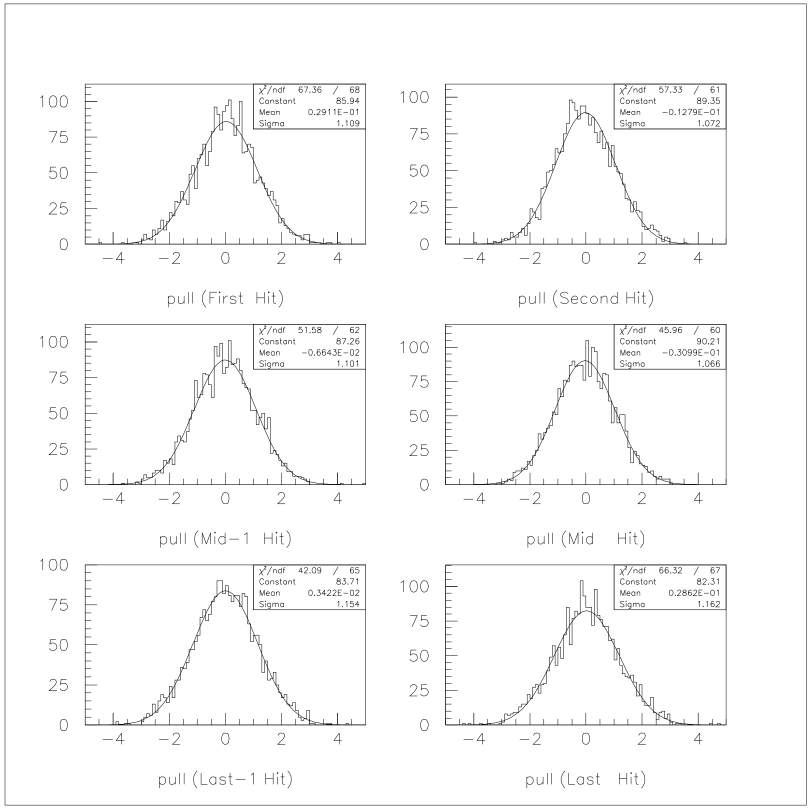

Figure 3: Pull distributions for six representative hits. The six representative hits are chosen to be first, second, middle-1, middle, last-1, and last hits. For even number of hits, middle hit is taken to be nhit/2+1.

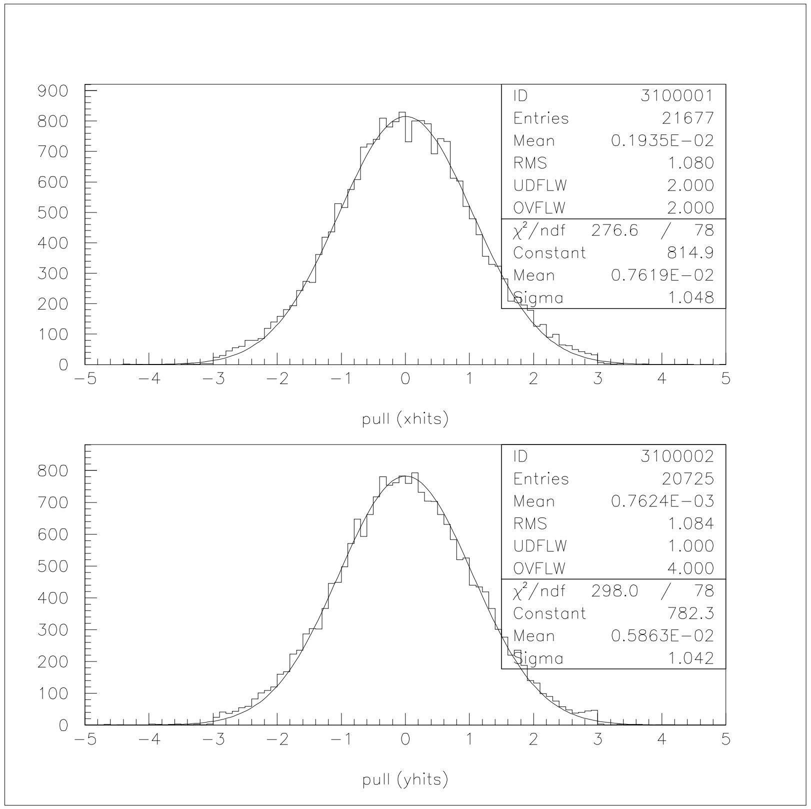

Figure 4: Pull distributions for all X-Z hits (Top plot) and all Y-Z hits (Side plot).

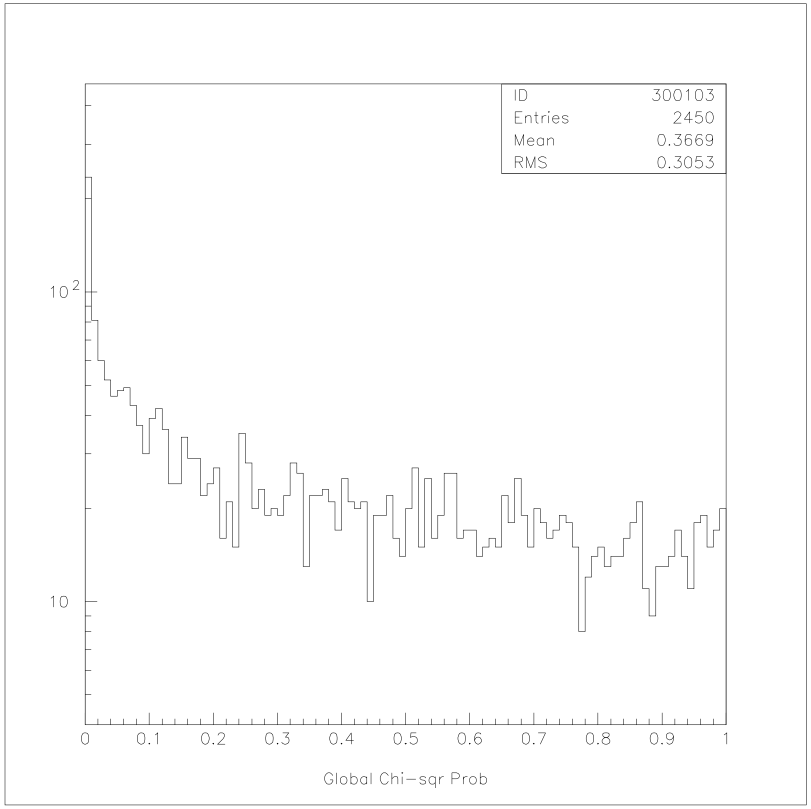

Figure 5: The global χ^2 probability distribution of a track.

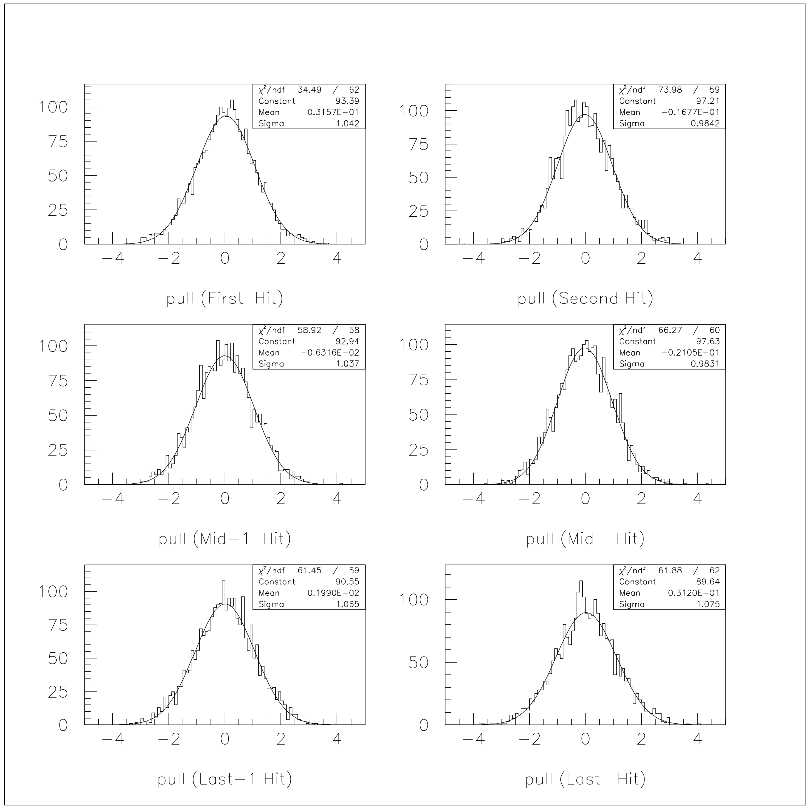

Figure 6: Pull distributions for six representative hits after the Aluminium thickness is modified from 0.36cm to 1.5cm between one X-Z section and Y-Z section. The six representative hits are chosen to be first, second, middle-1, middle, last-1, and last hits. For even number of hits, middle hit is taken to be nhit/2+1.

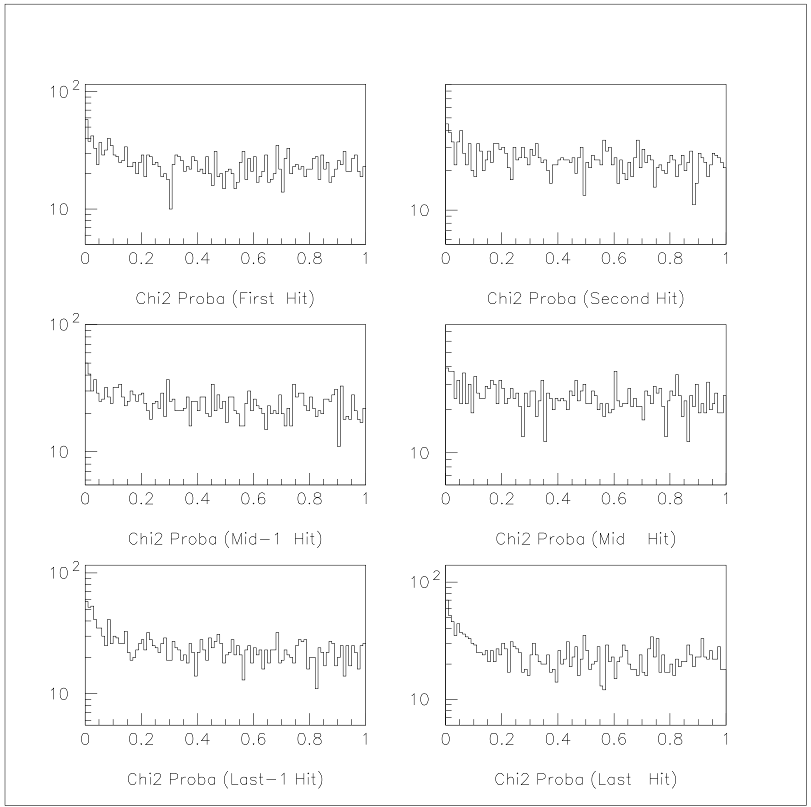

Figure 7: The individual χ^2 probability distribution for the six representative hits in a track. The χ^2 is calculed after 'Smoothing'.

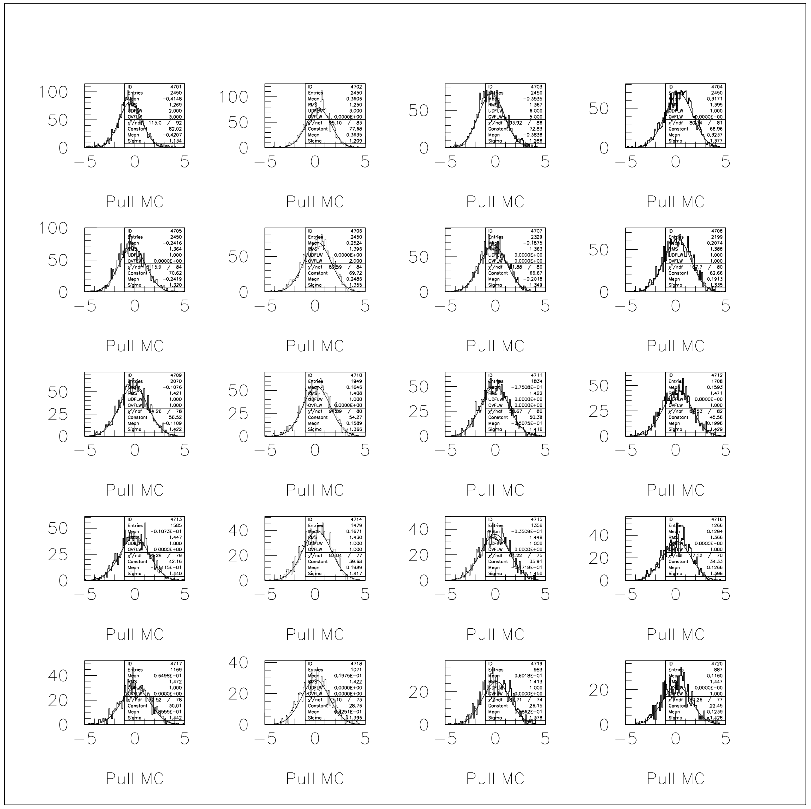

Figure 8: Distributions of pulls on r_k^{fit} for the first 20 hits.

For Gaussian measurement errors, distribution of the variable is Gaussian with rms equal to one with its mean zero.

- Tests on the Fake tracks

Get a reconstructed track (slopes and intercepts)

Randomize the scattering angles with the Gaussian of width θ_0

Calculate changes of slopes and intercepts

Reconstructed hits are simulated by smearing the intrinsic resolution used in the MC events.

- Cosmic Ray Data

Assume a muon of 1 GeV in momentum

Intrinsic Resolution = MC intrinsic resolution

No software alignment

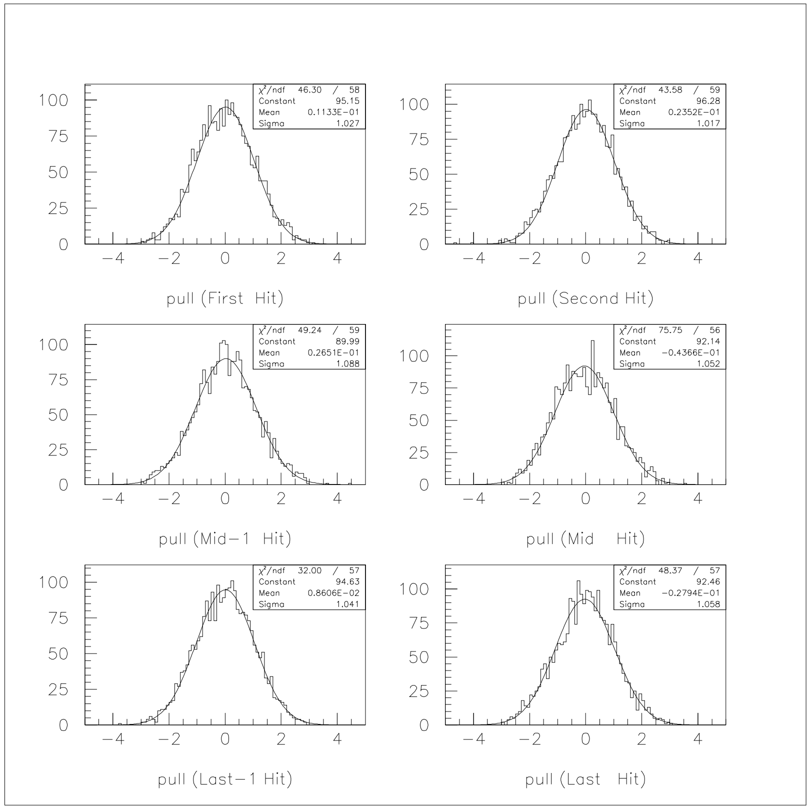

Figure 9: Pull distributions for six representative hits in a fake track. The six representative hits are chosen to be first, second, middle-1, middle, last-1, and last hits. For even number of hits, middle hit is taken to be nhit/2+1.

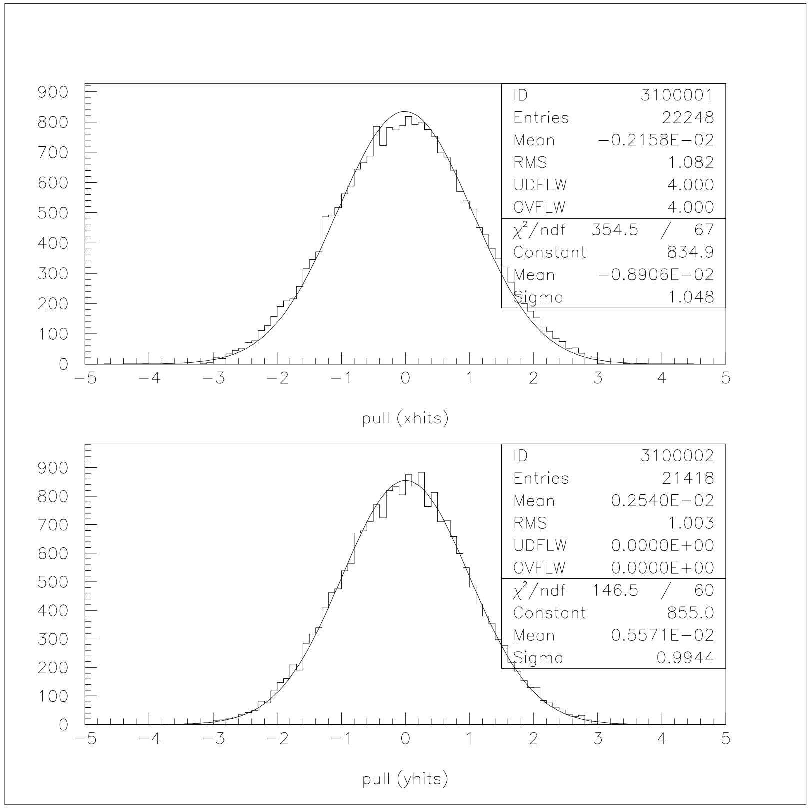

Figure 10: Pull distributions for all X-Z hits (Top plot) and all Y-Z hits (Side plot) in a track.

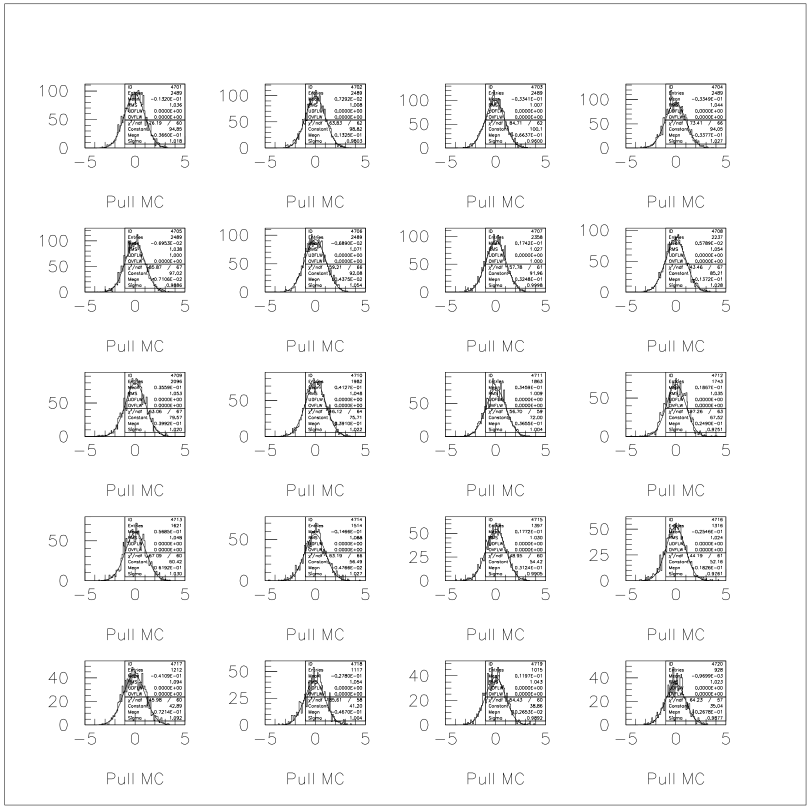

Figure 11: Distributions of pulls on r_k^{fit} for the first 20 hits in a fake track.

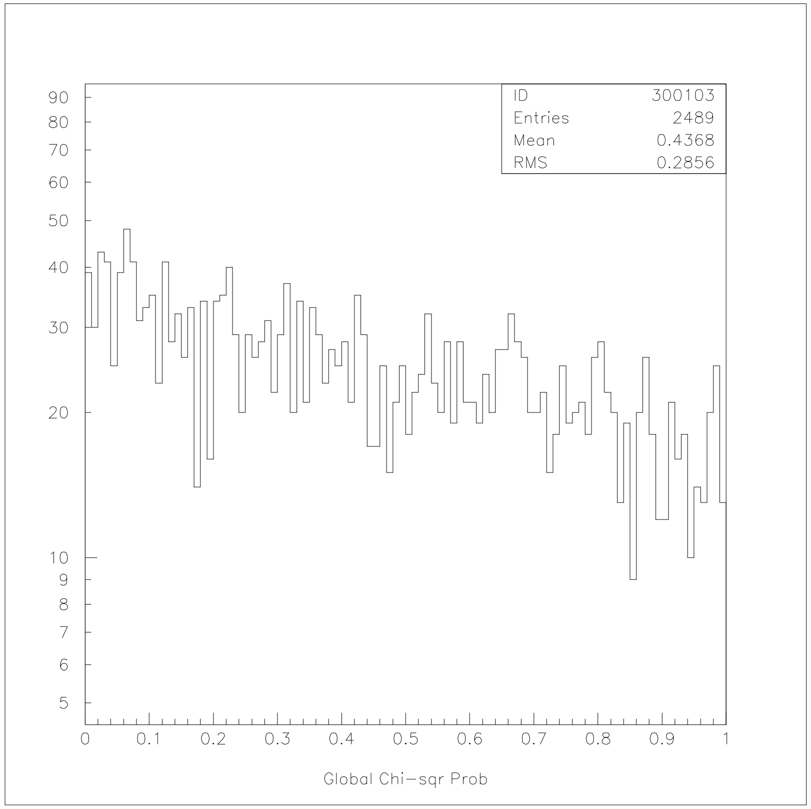

Figure 12: The global χ^2 probability distribution of a fake track.

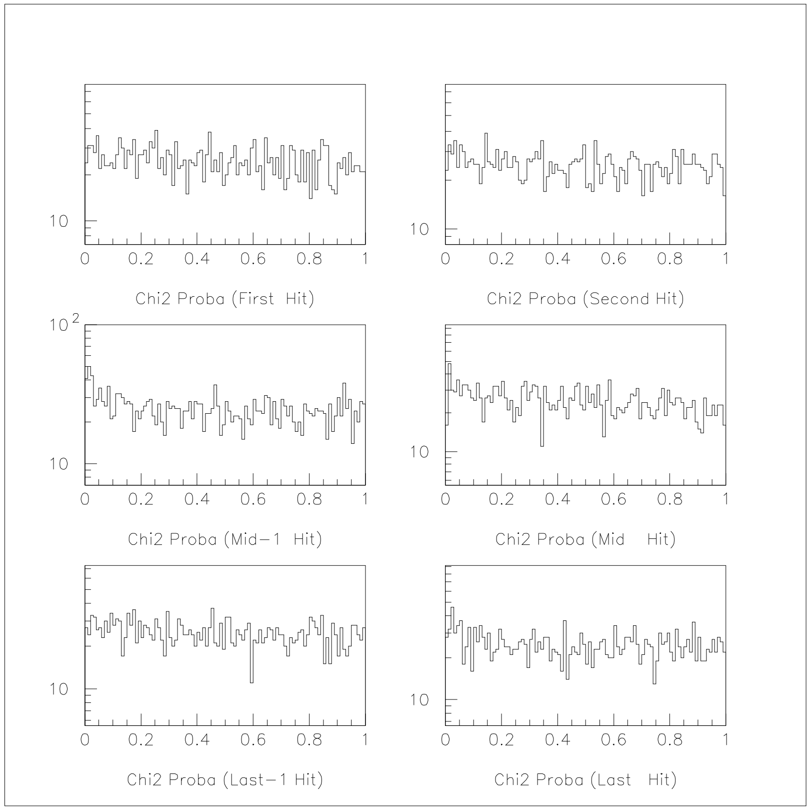

Figure 13: The individual χ^2 probability distribution for the six representative hits in a fake track. The χ^2 is calculed after 'Smoothing'.

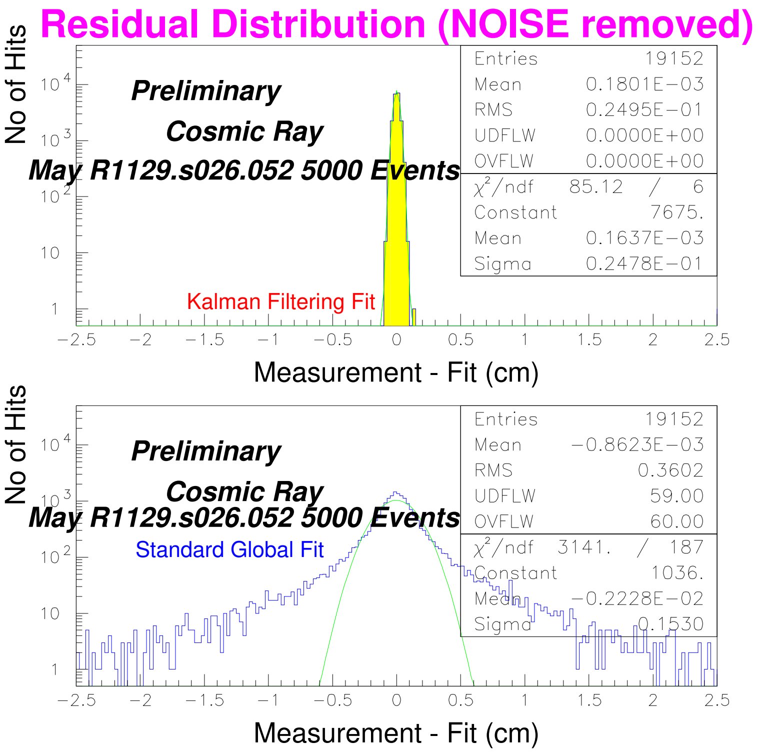

Figure 14: A comparison of residual distributions between the Kalman filtering fit and the standard global fit for May 1999 cosmic ray data.

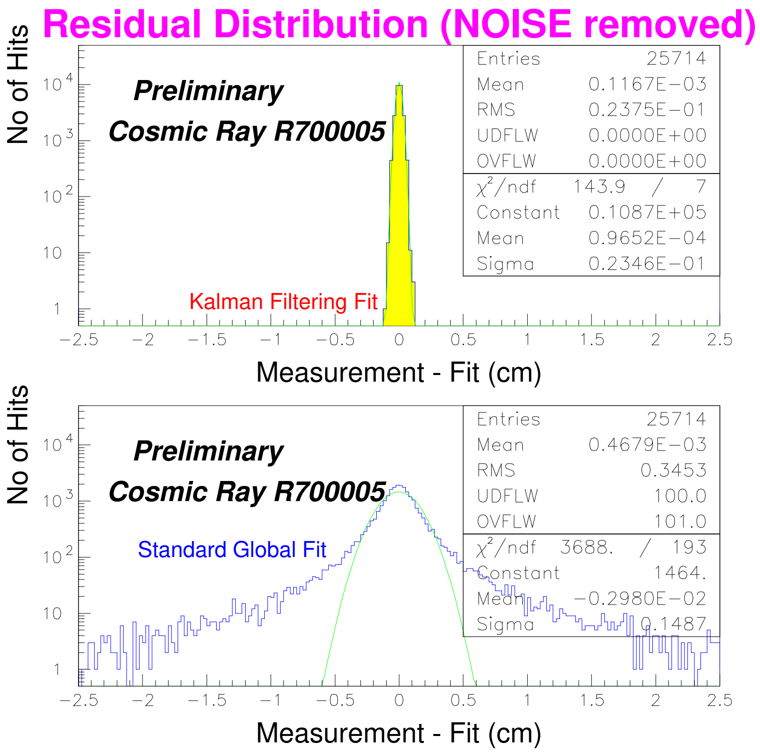

Figure 15: A comparison of residual distributions between the Kalman filtering fit and the standard global fit for July 1999 cosmic ray data.

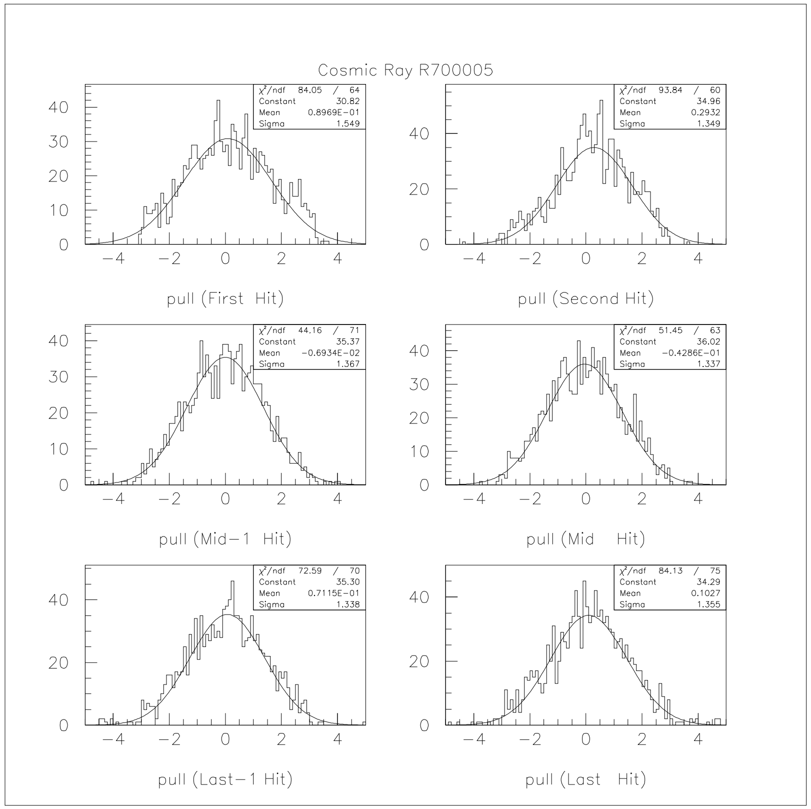

Figure 16: Pull distributions for six representative hits for the July 1999 cosmic ray data. The software alignment was not performed. The intrinsic resolutions were assumed to be same as the MC events with the muon momentum of 1 GeV.

KALMAN FILTERING VARIABLES FOR BANKS

the global χ^2

the individual pulls after smoothing

the global χ^2 (π or P assumption)

Noise hits not used in the Kalman Filtering

SUMMARY

Implementation of Kalman Filtering explained

MC events show consistency between the algorithm and the pulls

Cosmic Ray Data needs Alignment correction

The public version for test will be released in a day or two

The implementation can be extended to other detectors